

# Modified Gravity

*Léo Vacher*

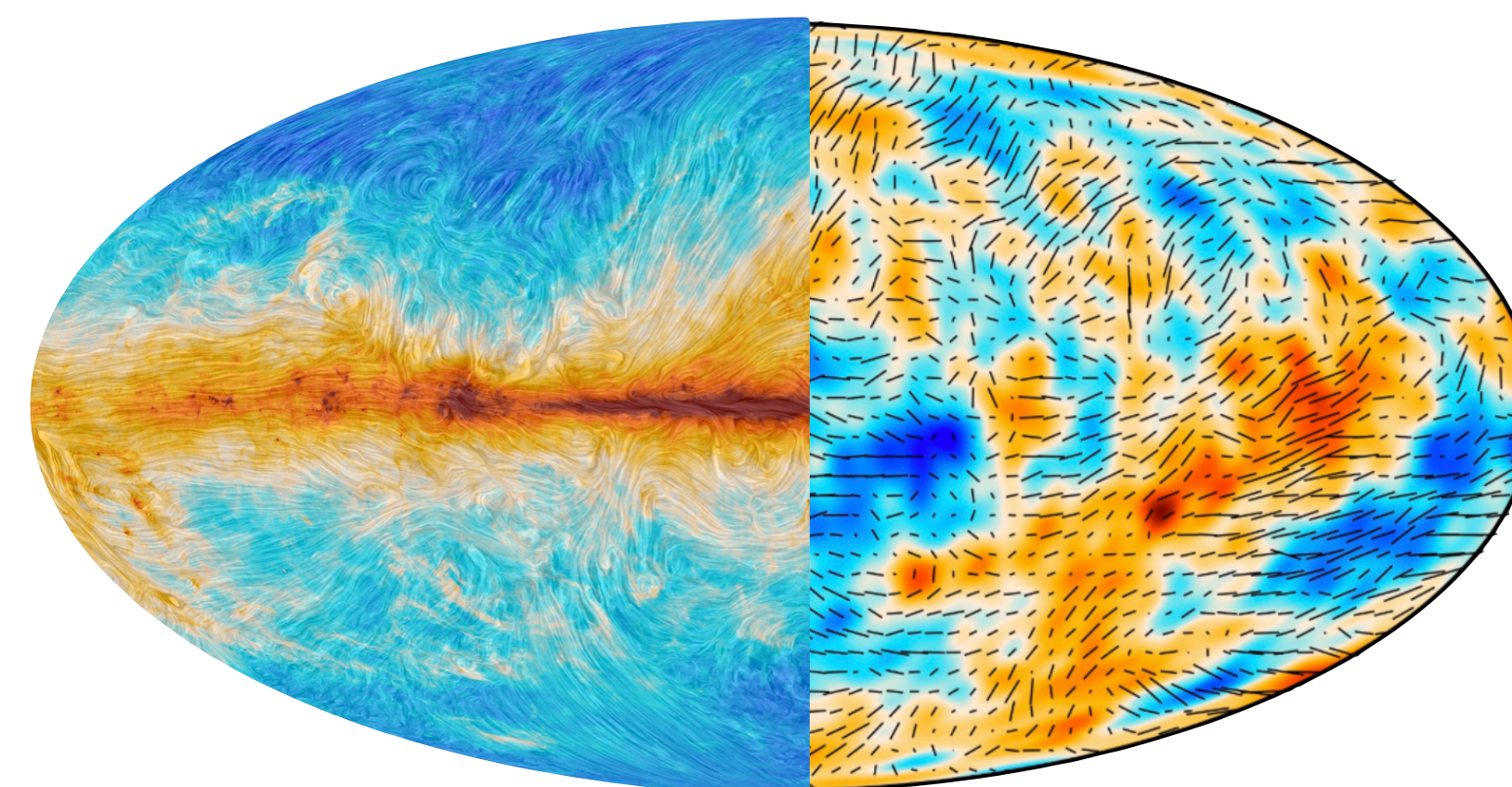


# My research

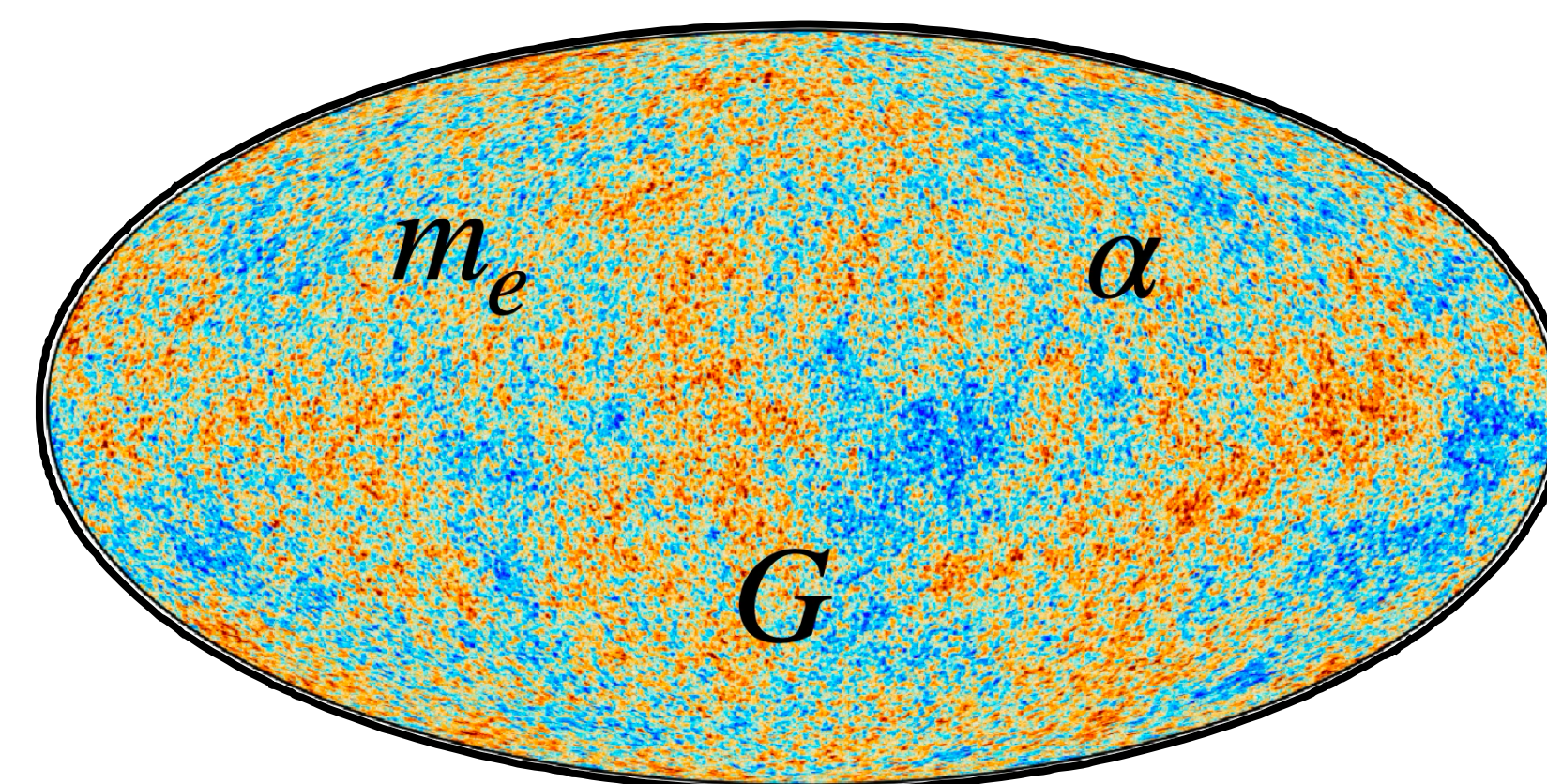
- I am a **chargé de recherche CNRS** at **IJCLab** (Orsay, south of Paris)
- I did my PHD at **IRAP**, (Toulouse) (end 2023) and a postdoc at **SISSA** (Trieste, Italie)
- I work mostly on **CMB science** (its polarisation) as well as a bit of **history and philosophy of physics** (mostly gauge theories)
- Contact me at [leo.vacher@ijclab.in2p3.fr](mailto:leo.vacher@ijclab.in2p3.fr)



**Component separation/modelling of the Galactic signal**



**Measure of the fundamental constants in the early universe (and elsewhere!)**





# This class

General gravity is a **beautiful**, and also **extremely resilient** theory:

- Is General relativity our final **theory of gravitation**?
- **Why and how** can we go beyond?
- How **cosmology** can probe the nature of gravity?

I think I have been way too ambitious! We'll go where we can go!



# This class

- I - Foundations of **general relativity: theory and observations**
- II - theoretical models of **modified gravity**
- Hands-on **Brans-Dicke**, the archetypal scalar field model (?)



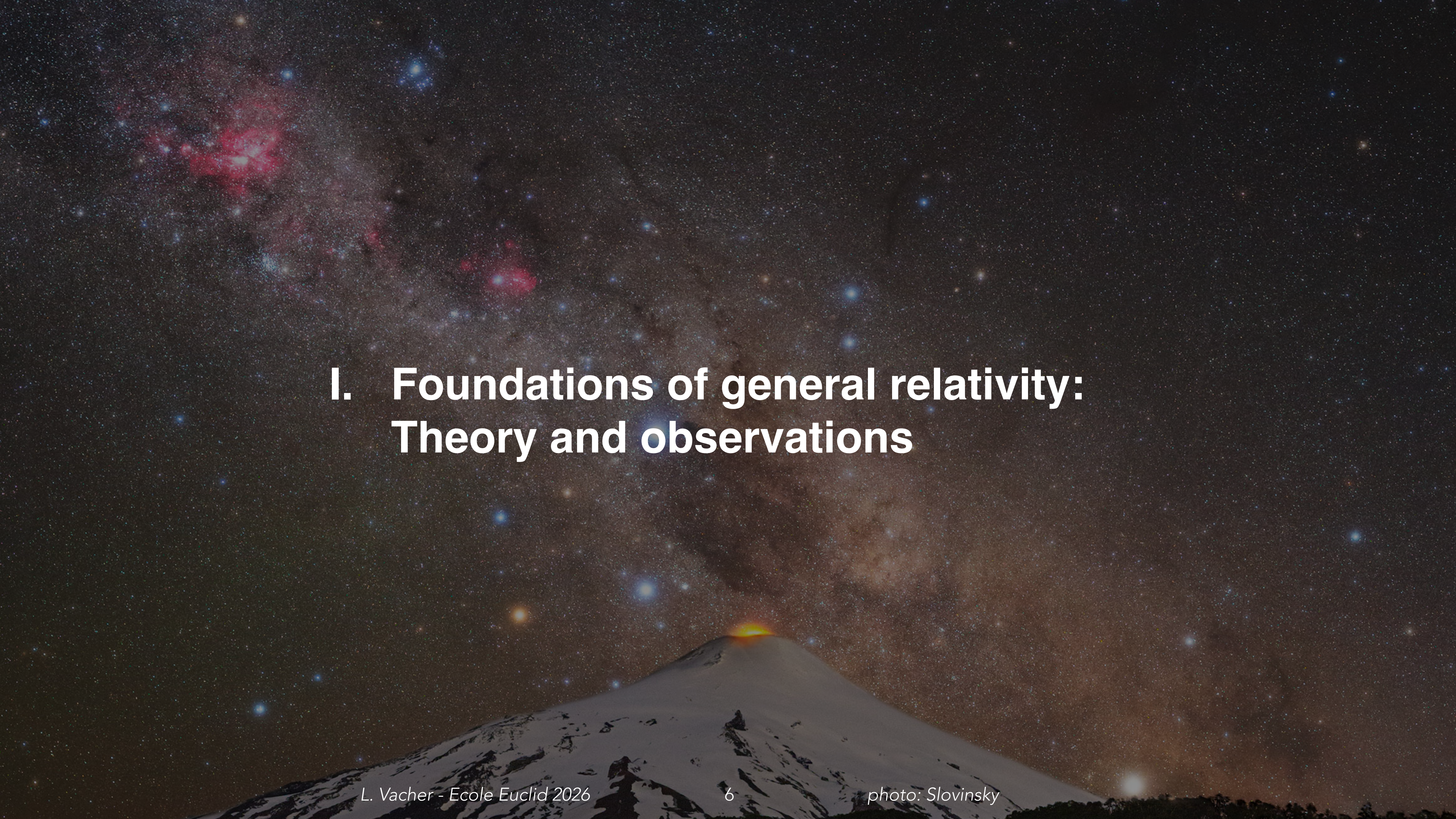
Lecture notes and proofs available at:

[https://yolonomy.github.io/cosmo/modified\\_gravity/mod\\_grav/](https://yolonomy.github.io/cosmo/modified_gravity/mod_grav/)



Incomplete and probably a lot of typos!!!! Let me know if you spot some



The background of the slide is a composite image. The lower portion shows a snow-covered mountain peak, likely Mount Fuji, with a small, bright orange-yellow light source (possibly the sun or moon) just above its summit. The upper portion of the image is a deep space photograph showing the Milky Way galaxy in great detail, with its characteristic band of stars and dust stretching across the sky. Several bright, distinct stars are visible against the dark background of space.

# **I. Foundations of general relativity: Theory and observations**

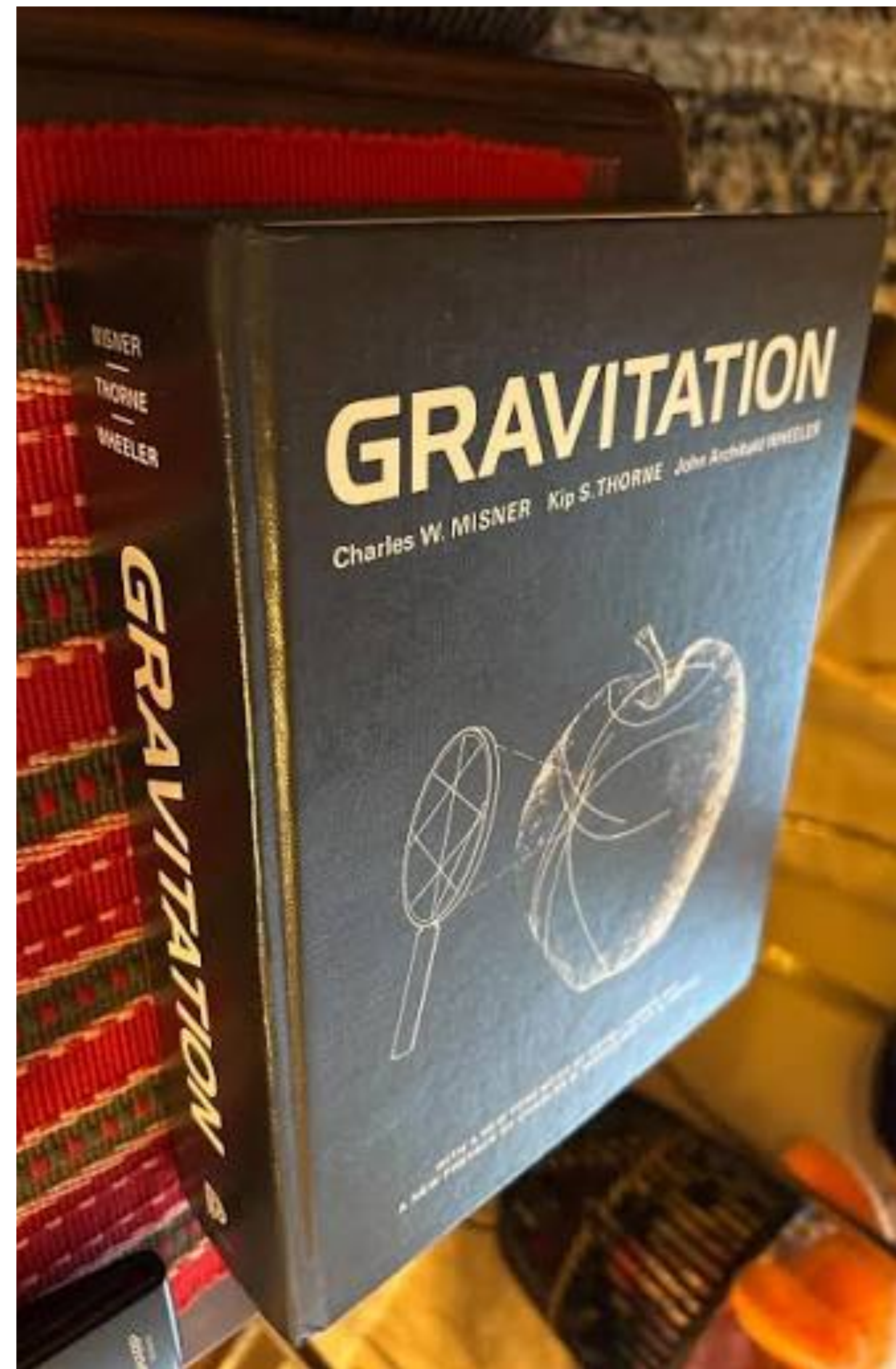


# To study modified gravity, you must have a deep understanding of GR

**Gravitation:** [\*Misner, Thorne and Wheeler\*](#):

- 2,72 kg
- 1336 pages
- 1973 (needs 2000 more pages)

Another recommendation: [\*Will \(2018\)\*](#) (see notes)





# What to ask of a gravity theory ?

At minima, **gravity within a theory** should be described as:

- A **long range** force.
- Acting **between masses**.
- **Attractive**.
- Reproducing **Newton law** of gravitation under some suitable limits:  $V_{\text{int}} = -\frac{Gm_1m_2}{r}$   
with  $G$  **Newton gravitational constant**:  $G = 6.674300(15) \times 10^{-11} \text{m}^3\text{kg}^{-1}\text{s}^{-2}$  ([Codata 2022](#)).
- Reproducing **special relativity** when « turned off ».
- **Self-consistent** (no contradicting predictions).



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- **Self-consistent** (no contradicting predictions).

This might seem stupid but this constrain already a lot the possible theories.

**Newtonian gravitation** fails! (Why?), **GR satisfies** but it's **far from the only one**...



# The weak equivalence principle (WEP)

Now we can also ask for an additional requirement (should we?)

- Gravity is **universal**: it accelerates every mass identically.

This is an observational fact: the **Universality of free fall (UFF)**.

It can be used as a **principle** to build a theory (true both in Newt. and GR):

- **WEP**: all **free test** bodies fall identically in a gravitational field, independently of their **shape, state or composition**.



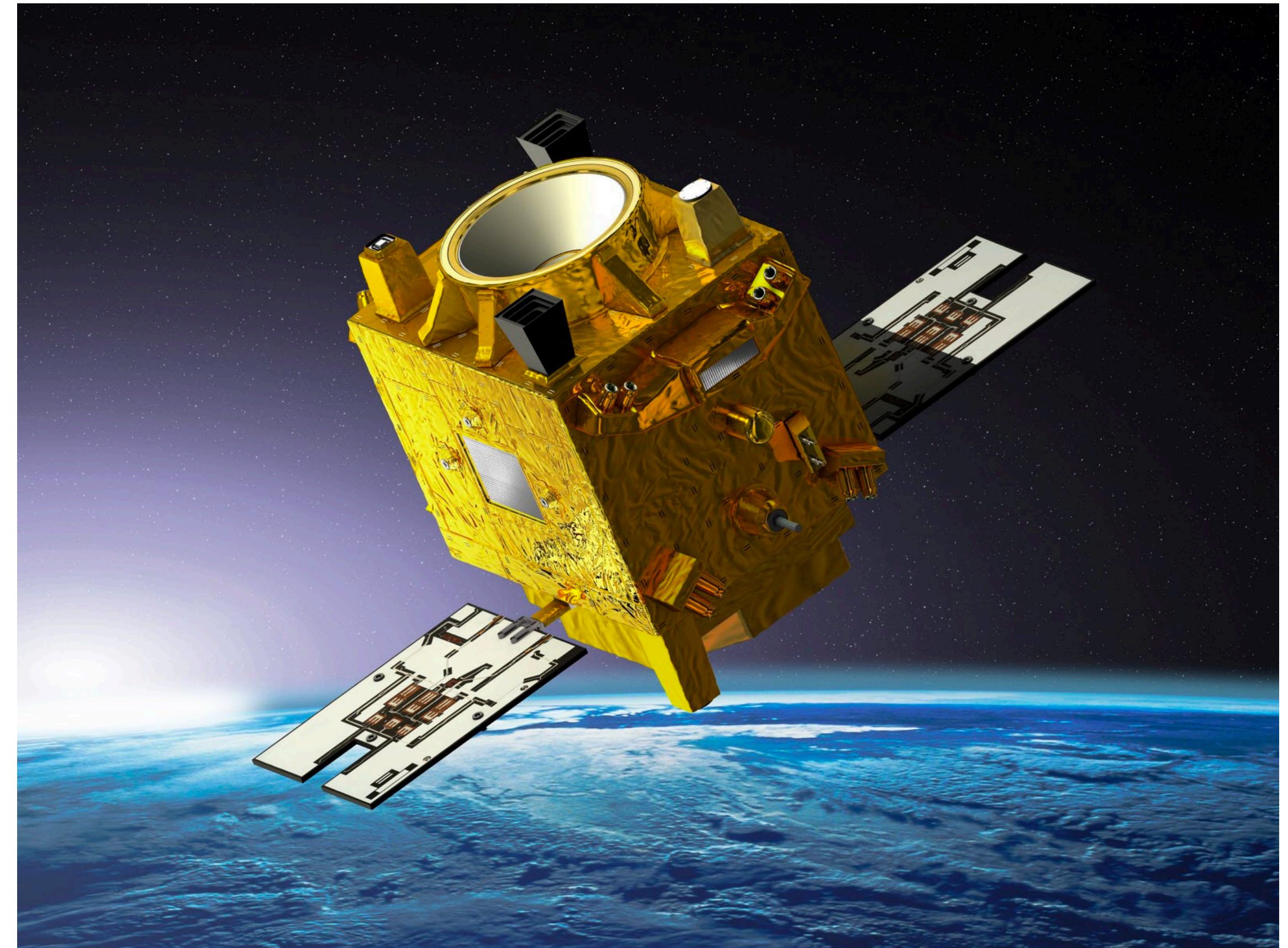
# The weak equivalence principle (WEP)

WEP tested by the **MICROSCOPE** mission ([Touboul+2021](#)) for a pair of titanium and platinum, the **Eötvös parameter**:

$$\eta = 2 \frac{|\vec{a}_1 - \vec{a}_2|}{|\vec{a}_1 + \vec{a}_2|}$$

Resulting in:

$$\eta = (-1.5 \pm 2.7) \times 10^{-15}$$



See also the review: [Bergé+ 2023](#)



# The Einstein equivalence principle (EEP)

We can add two other requirements on top of WEP, to build the EEP:

- **WEP** is true
- **Local position invariance (LPI)**: the outcome of any local non-gravitational experiment is independent of where and when in the Universe it is performed. As such, there is no “preferred” event in space-time in which laws of physics would be different.
- **Local Lorentz invariance (LLI)**: the outcome of any local non-gravitational experiment is independent of the velocity of the reference frame in which it is performed. Thus, inertial frames are all equivalent.



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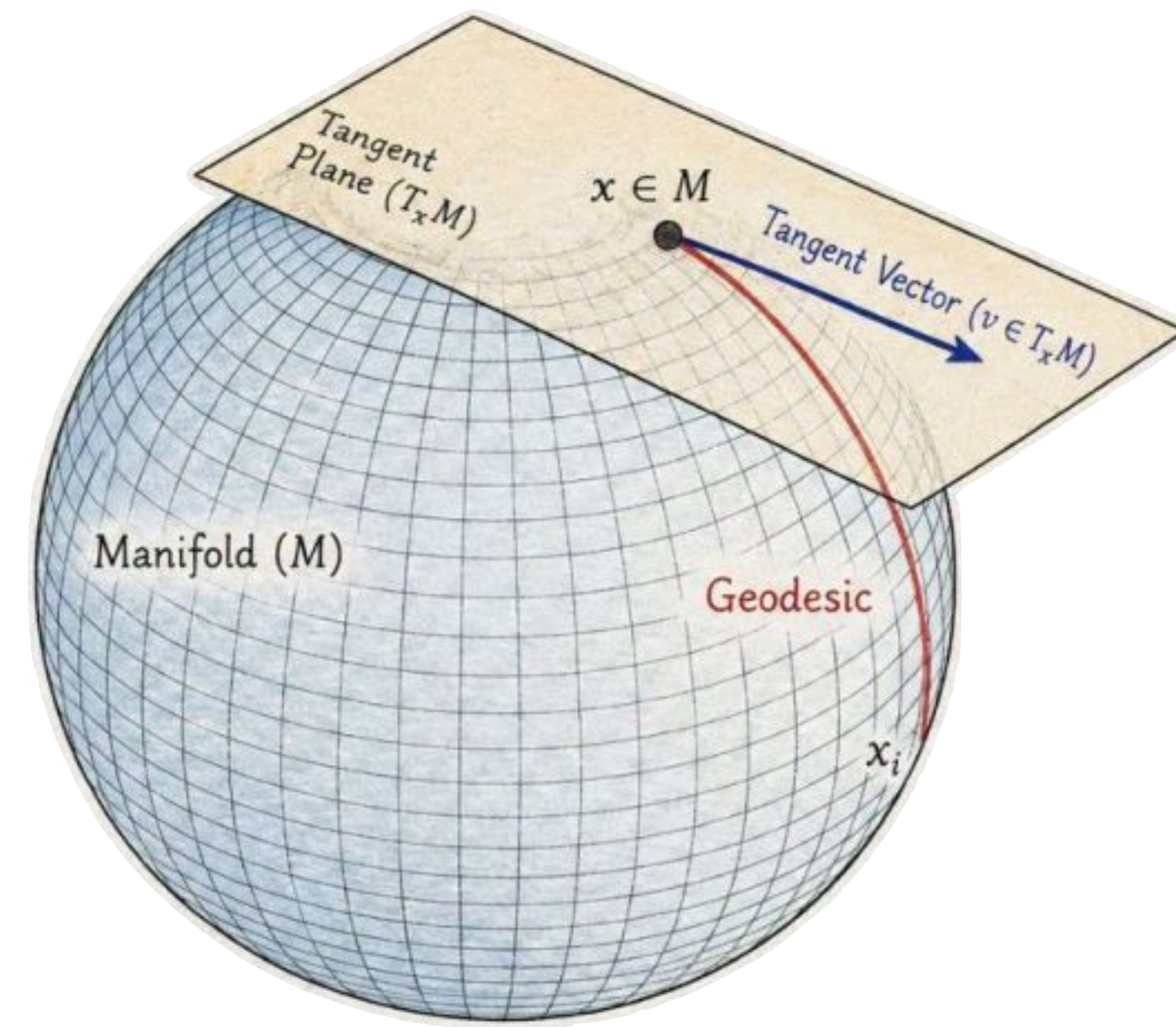
If one is true, we have reasons to believe that the three are: **Schiff's conjecture (see notes)**



# Metric theories of gravity

EEP allows the « **equivalence between acceleration and gravity** » that famously lead Einstein to GR. If it is valid, it allows for a **geometrization of gravity**:

- Space-time is a smooth **manifold**  $M$  equipped with a **Lorentzian metric**  $g$ ,
- WEP: Trajectories of freely falling test bodies are **geodesics** of the unique metric  $g$ ,
- At every point of space-time, it is possible to find a frame in which  **$g = \eta$**  (LPI + LLI). Free fall frames = inertial frames of special relativity

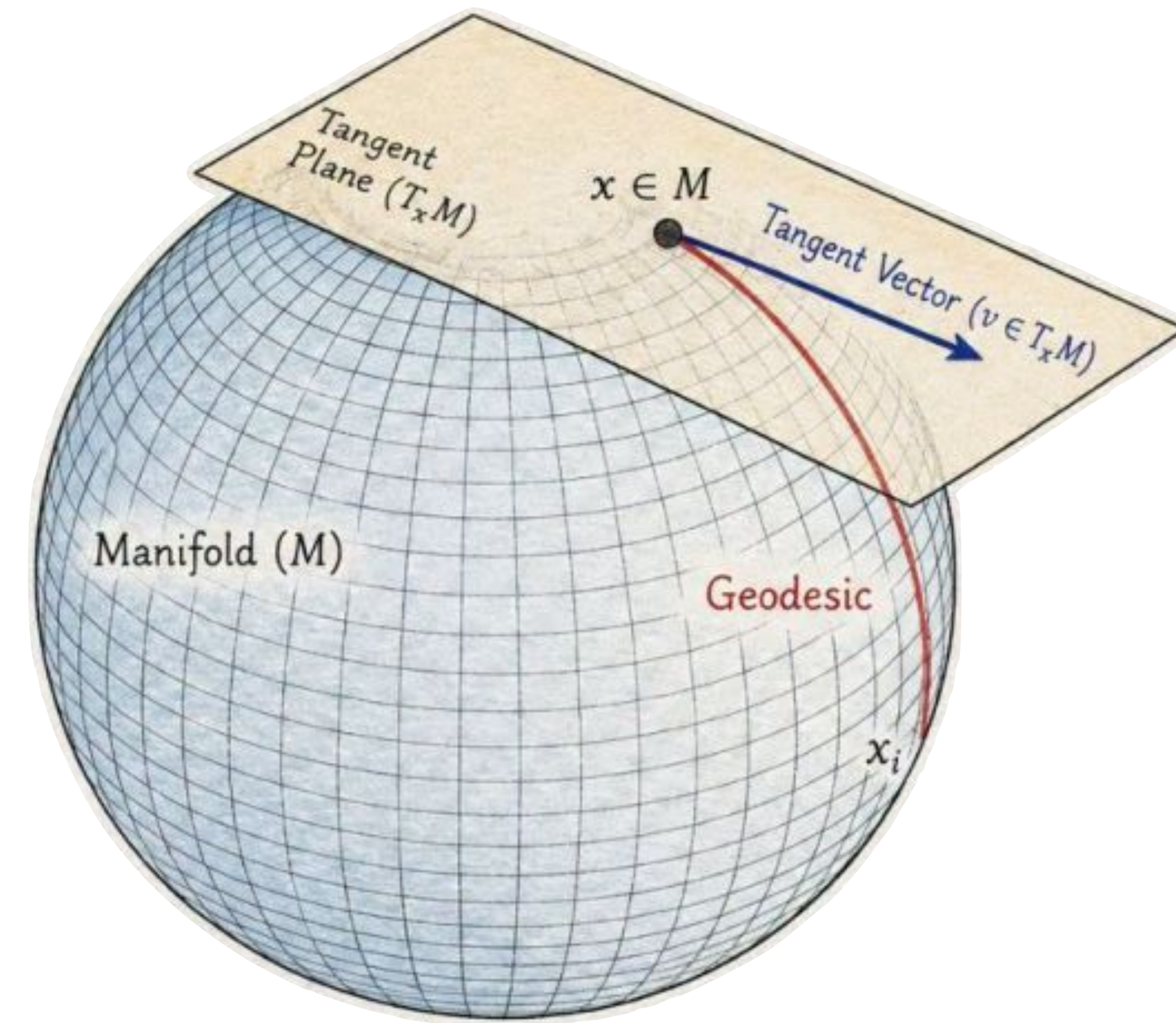




# Metric theories of gravity

EEP allows the « **equivalence between acceleration and gravity** » that famously lead Einstein to GR. If it is valid, it allows for a **geometrization of gravity**:

- Any theory satisfying the EEP is called a **metric theory of gravity**  
**GR is one but its not the only one!!!**
- Many theories **are non-metric** this is the case of **any formulation of string theory or loop quantum gravity** near Planck scale.
- EEP talks about **how matter respond to gravity** but nothing about how gravity responds to matter)

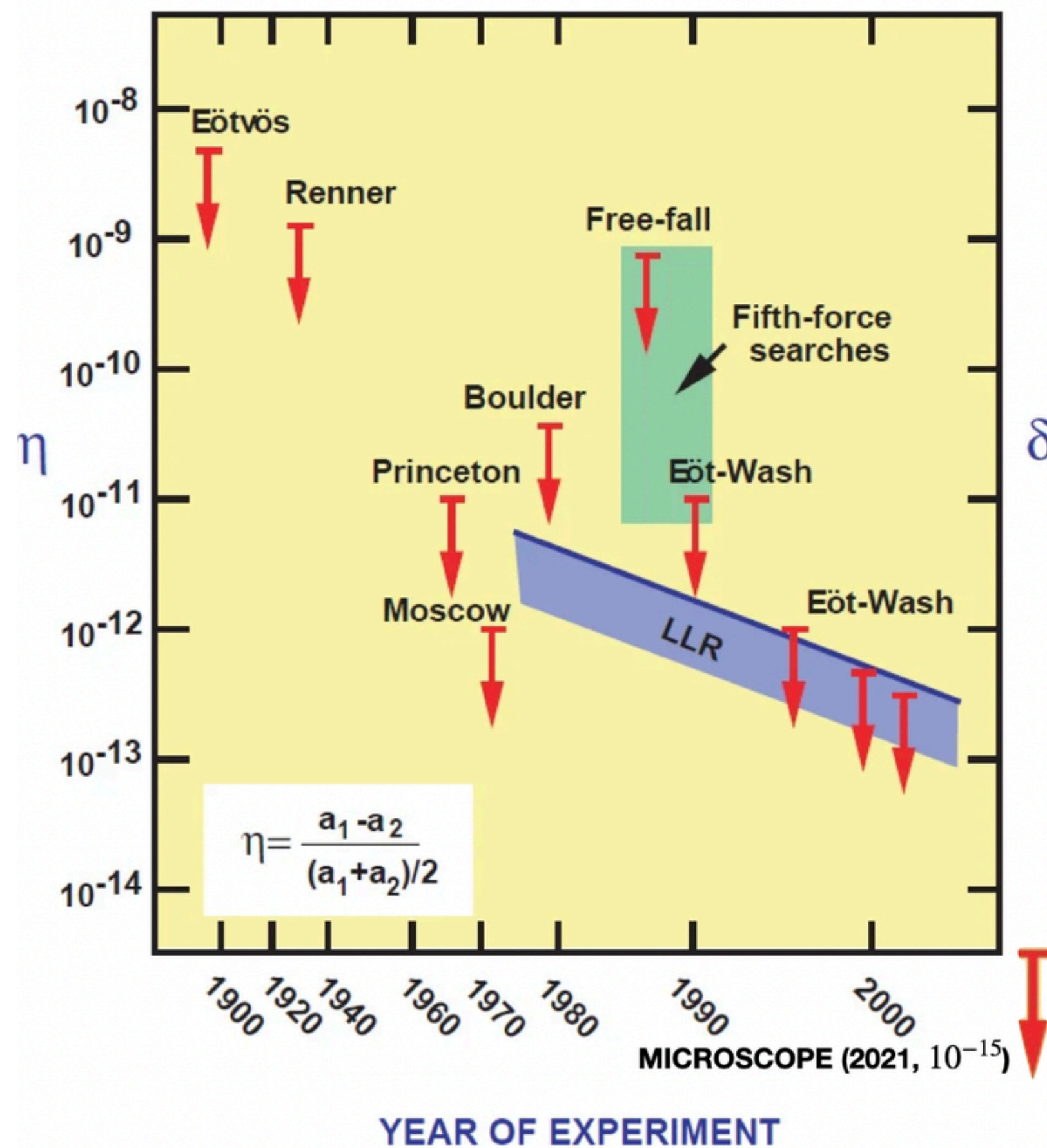




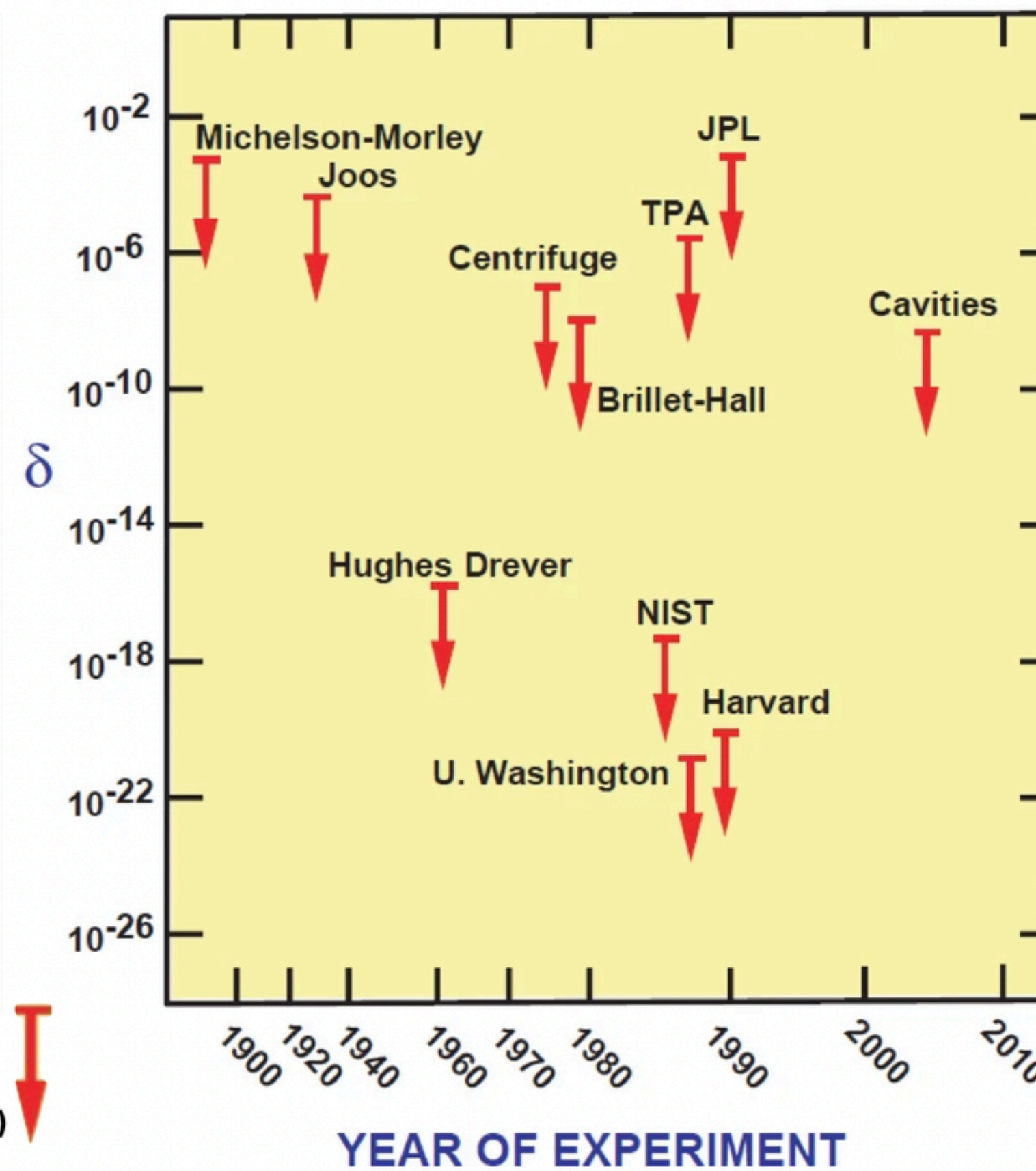
# Tests of the EEP: local

[Will \(2014,2018\)](#)

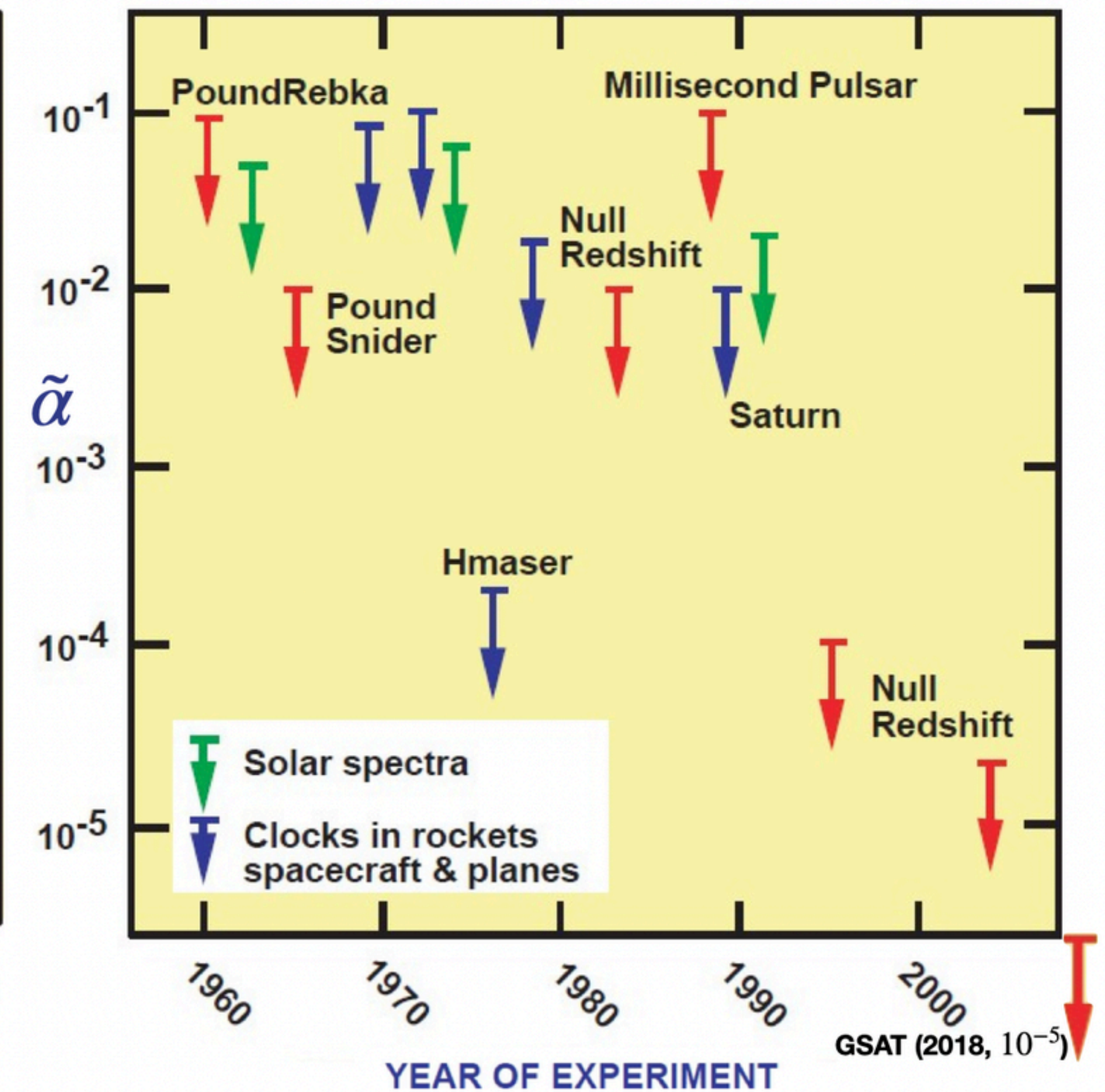
TESTS OF THE  
WEAK EQUIVALENCE PRINCIPLE



TESTS OF  
LOCAL LORENTZ INVARIANCE



TESTS OF  
LOCAL POSITION INVARIANCE





# Tests of the EEP: astrophysical

Testing for stability of (dimensionless) **non gravitational constants** as:

$$\alpha = \frac{e^2}{4\pi\hbar c} \sim \frac{1}{137} \quad \text{or} \quad \mu = \frac{m_e}{m_p} \sim \frac{1}{1840}$$

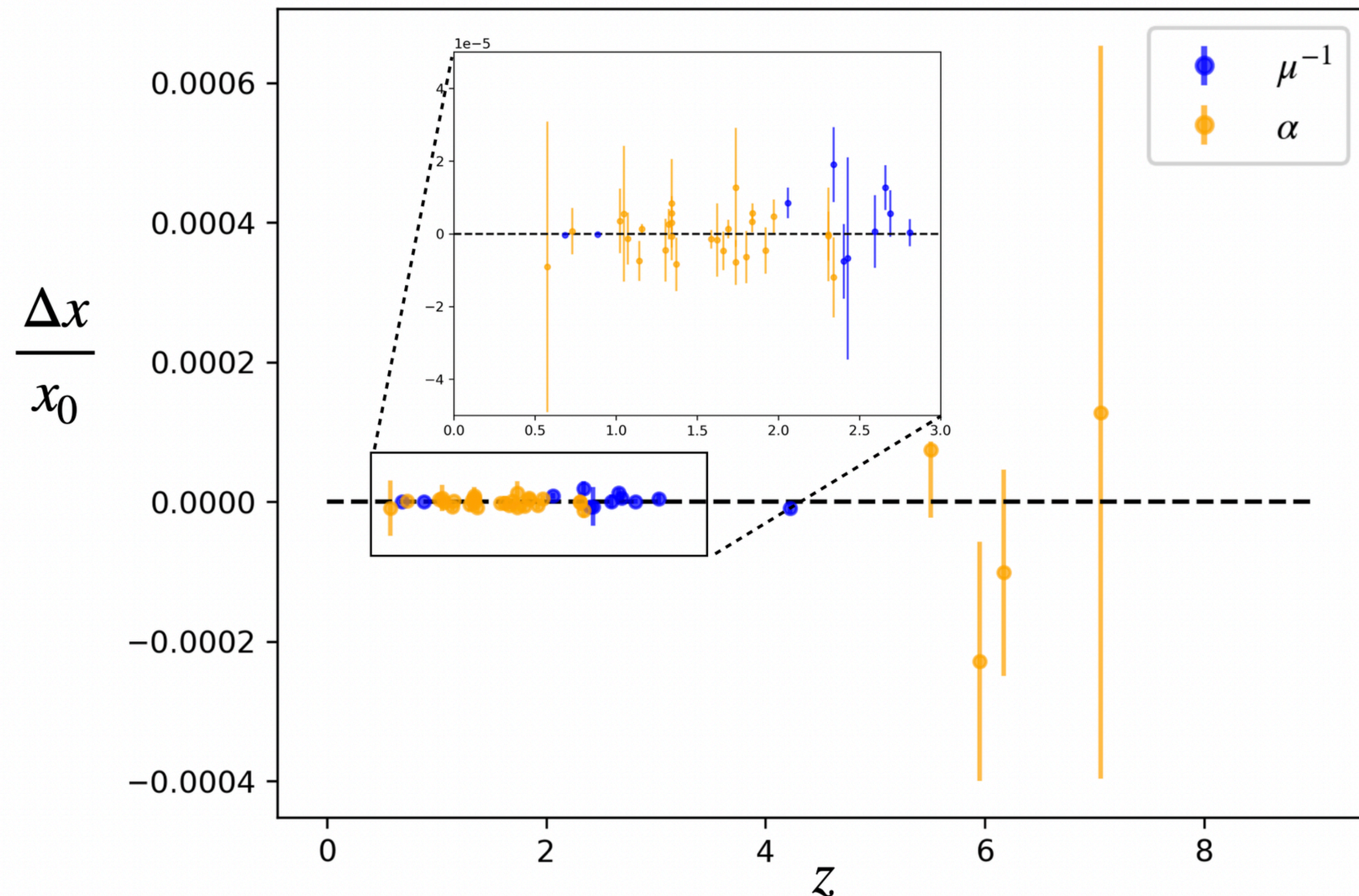
$$F_C = \frac{\alpha z_1 z_2}{r^2}$$

Is a direct test of the EEP (see notes)

Many complementary probes can be used (**laboratory, astrophysics, cosmology** ...)

$$u^\nu \nabla_\mu u^\mu = - \frac{d \ln m}{d \ln \alpha} \partial_\beta \alpha (g^{\beta\mu} + u^\beta u^\mu)$$

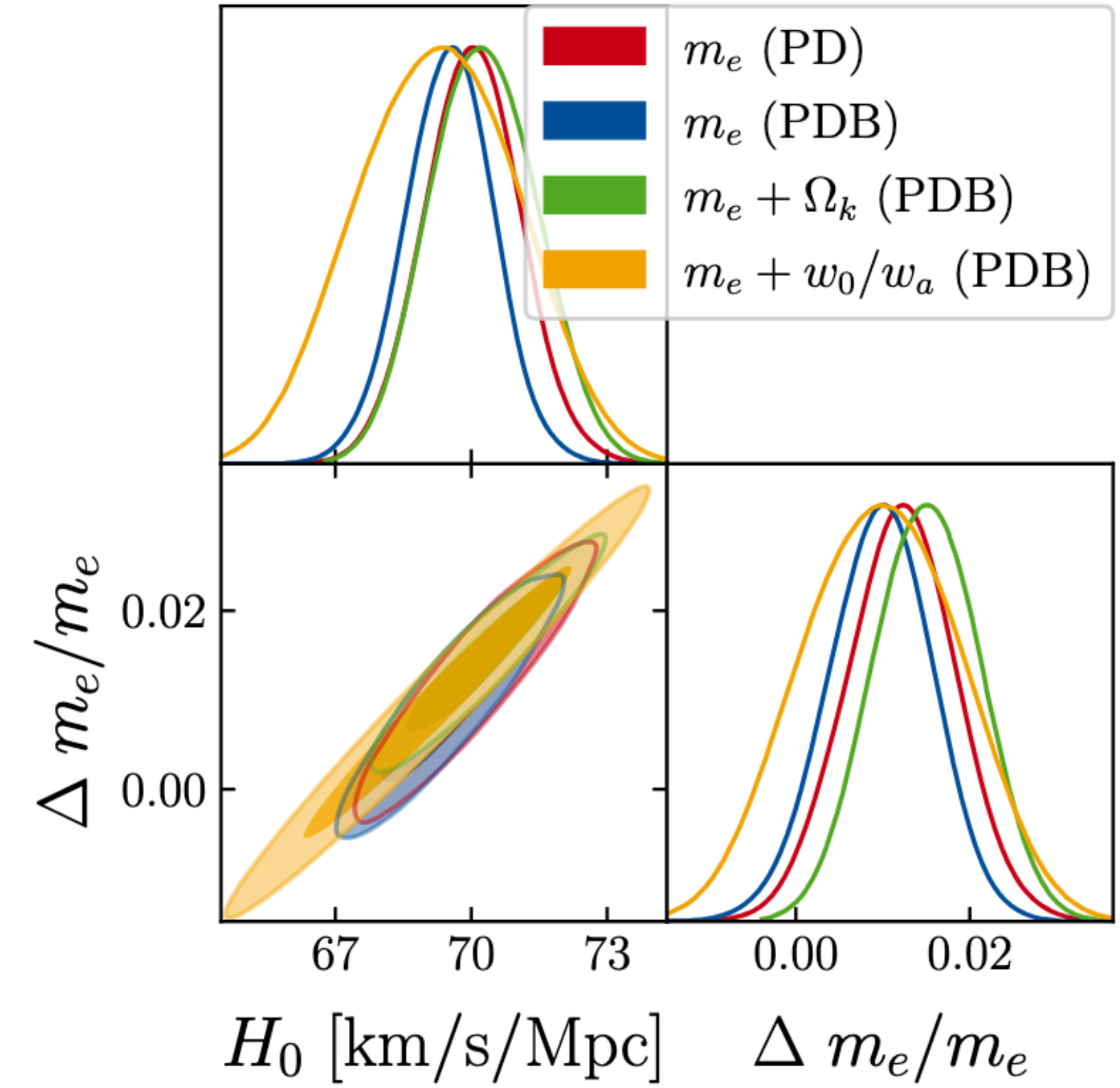
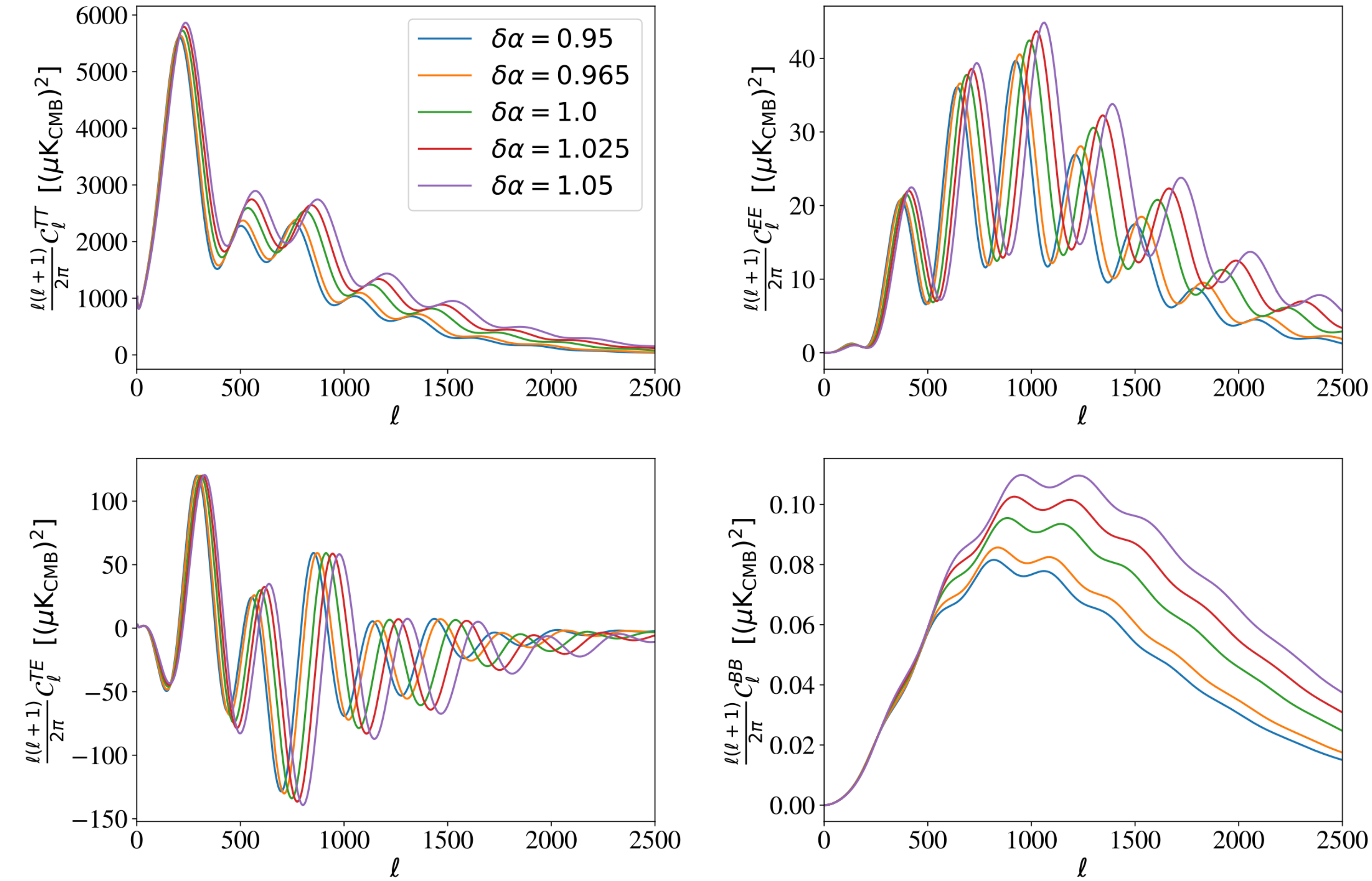
For a review see lecture notes and [Uzan \(2024\)](#)



My own plot with data collected in [Uzan \(2024\)](#)

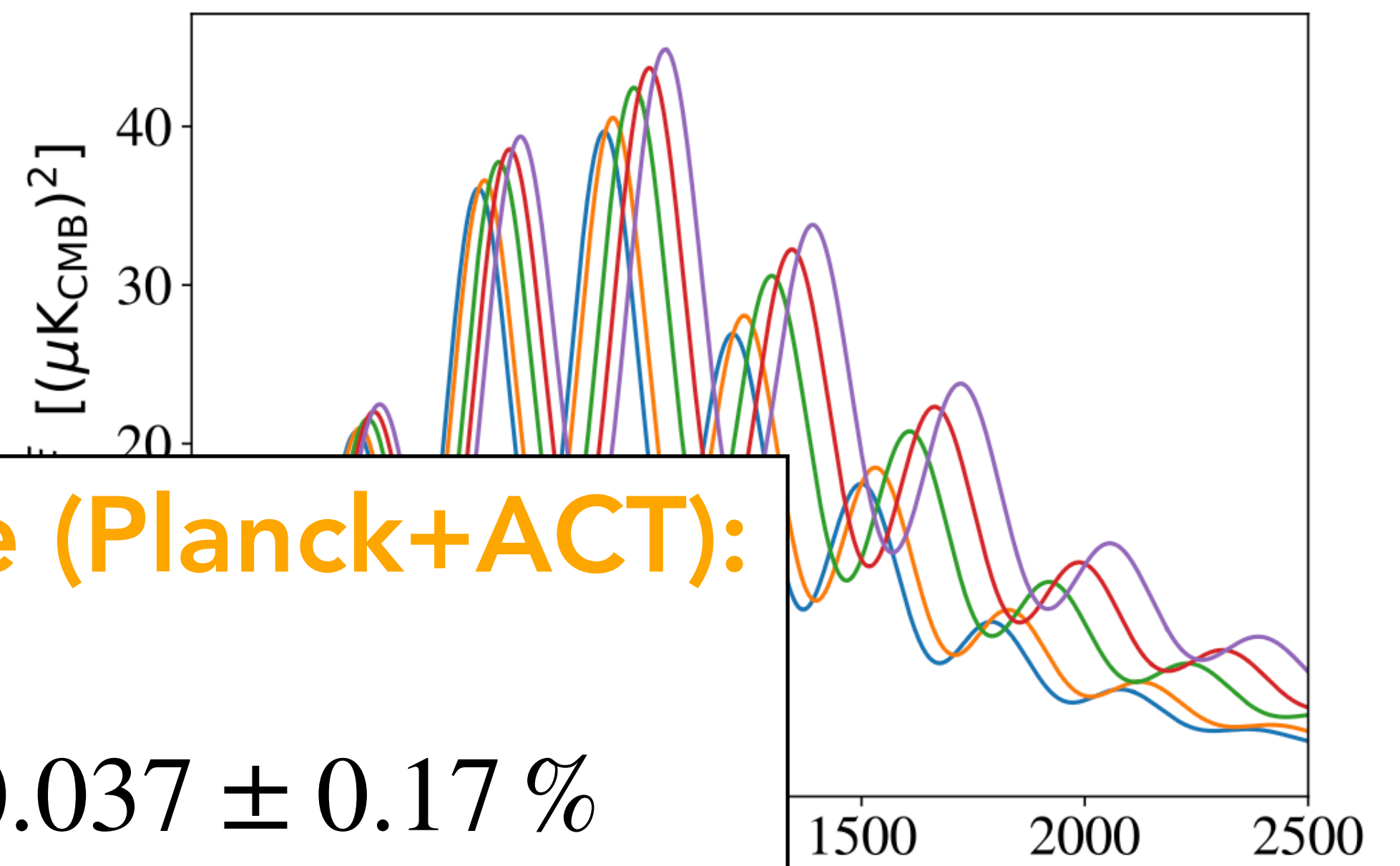
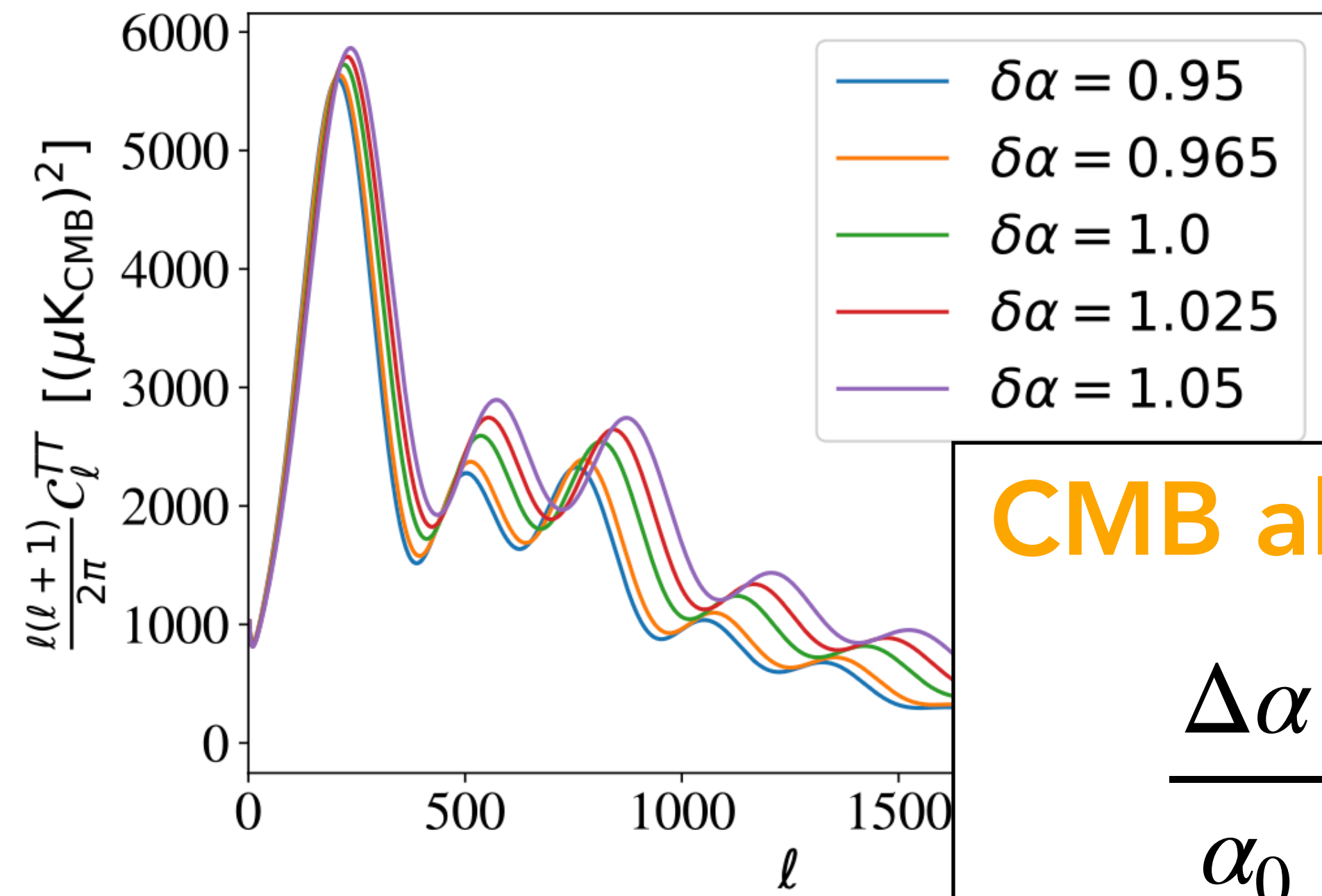


# Tests of the EEP: cosmological



[\(Schöneberg & Vacher 2024\)](#)



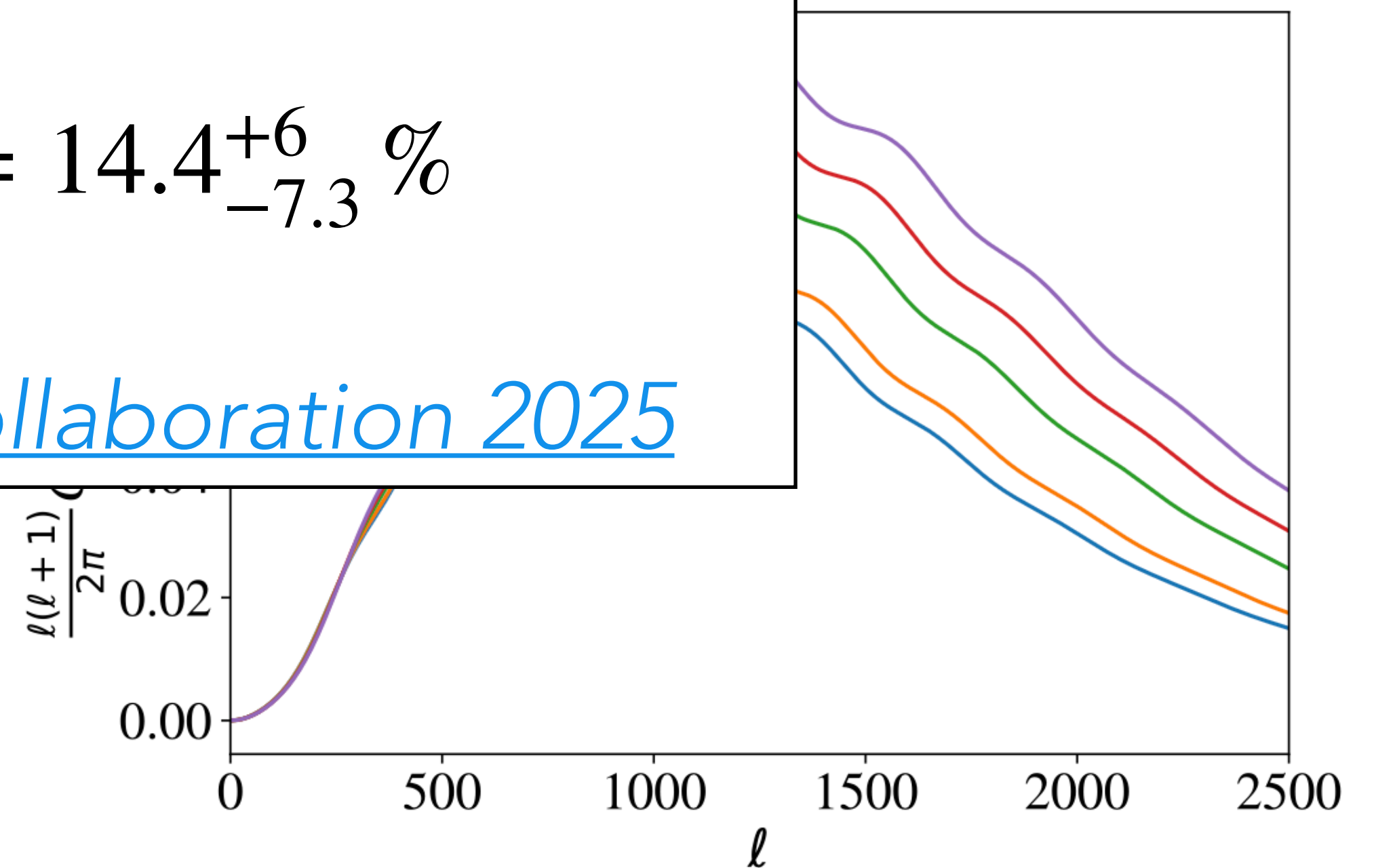
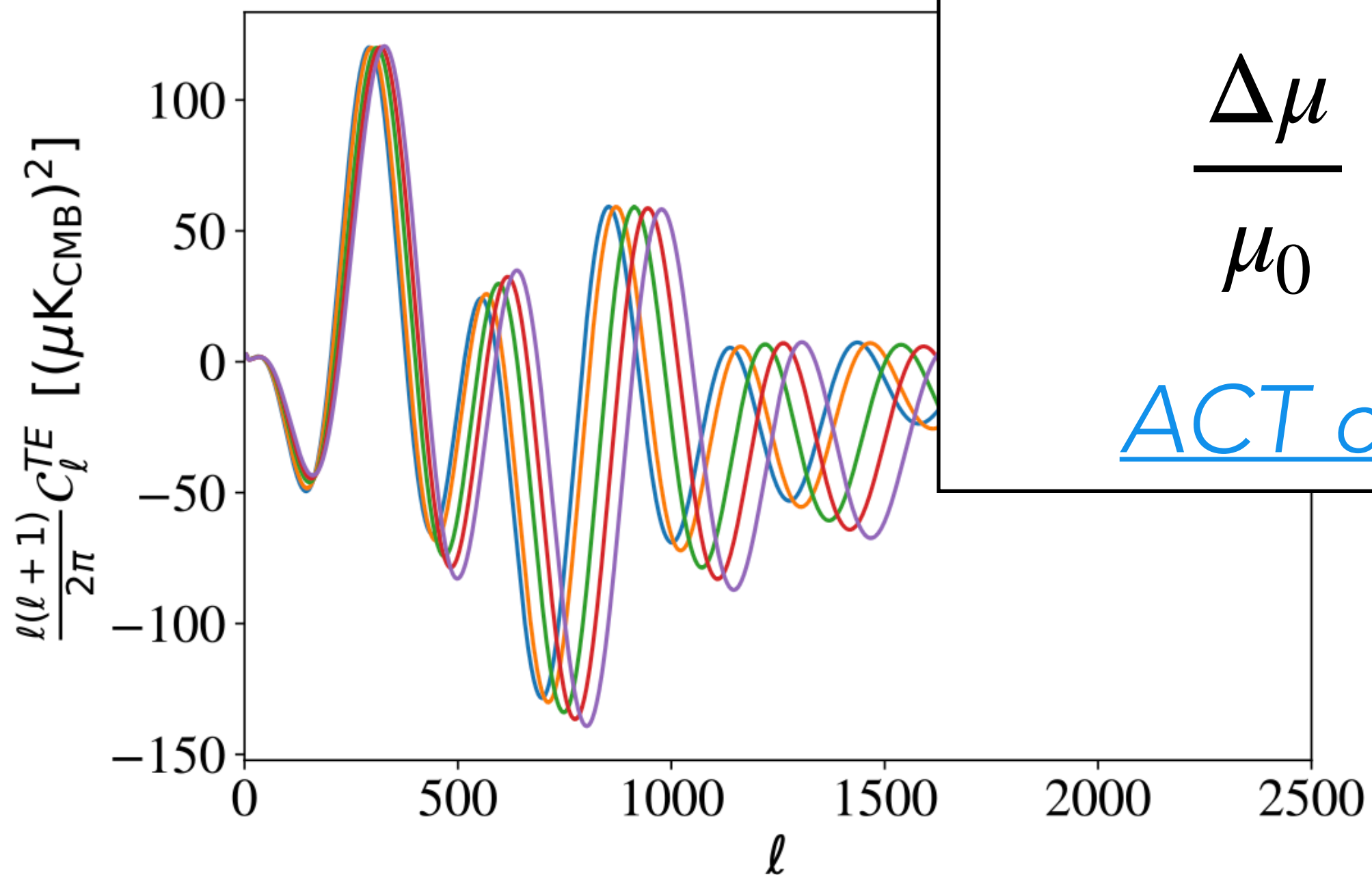


**CMB alone (Planck+ACT):**

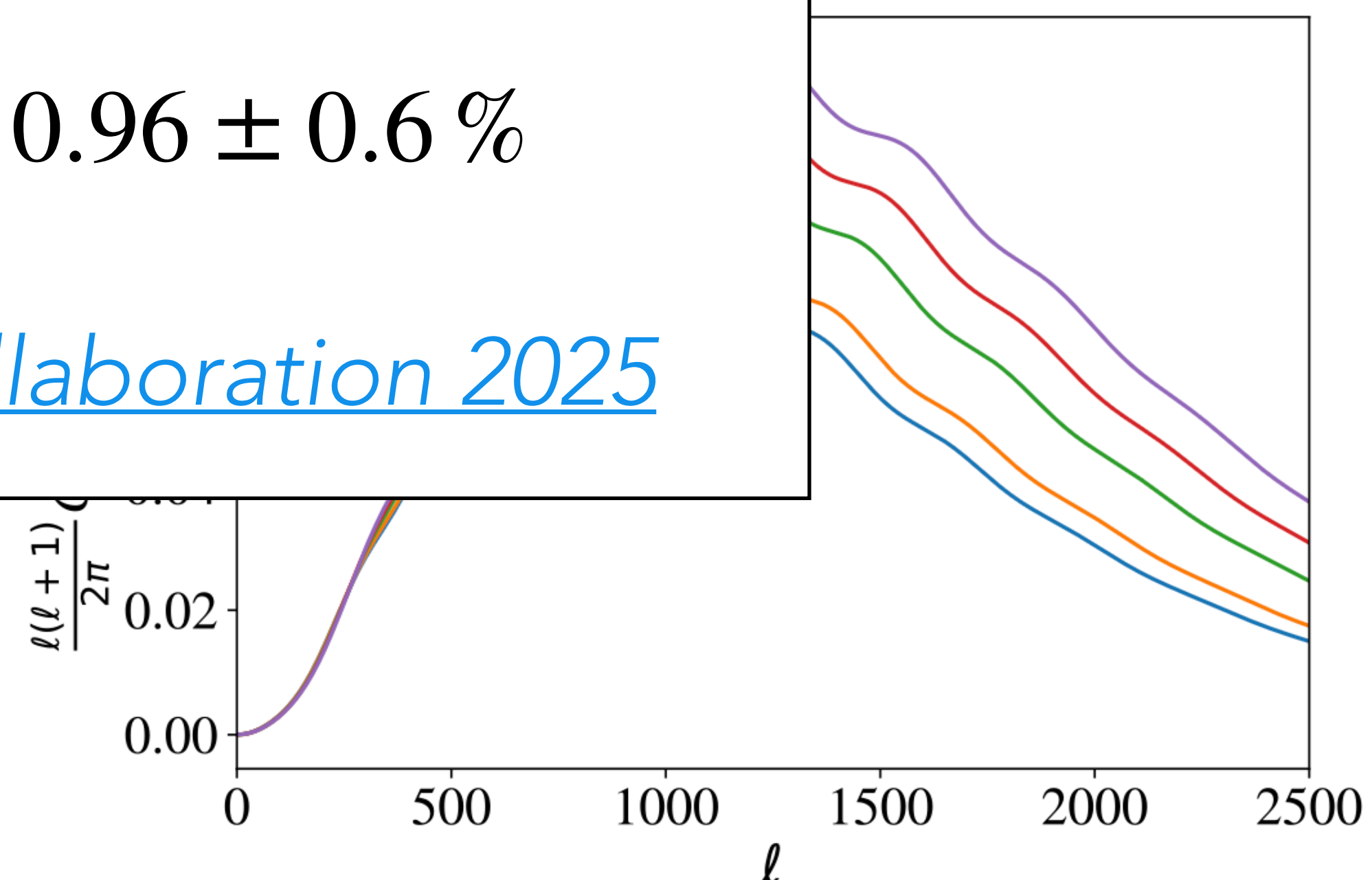
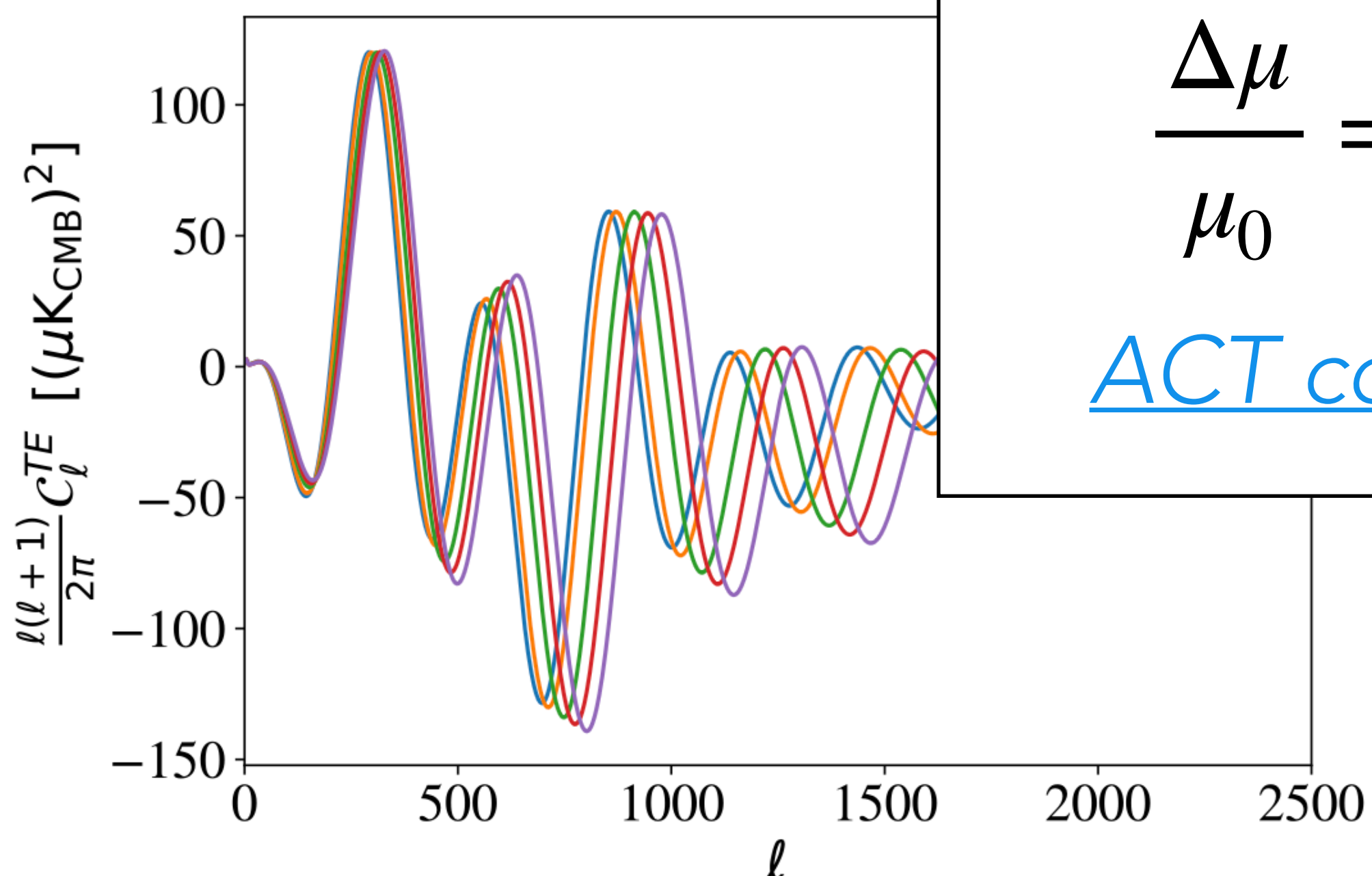
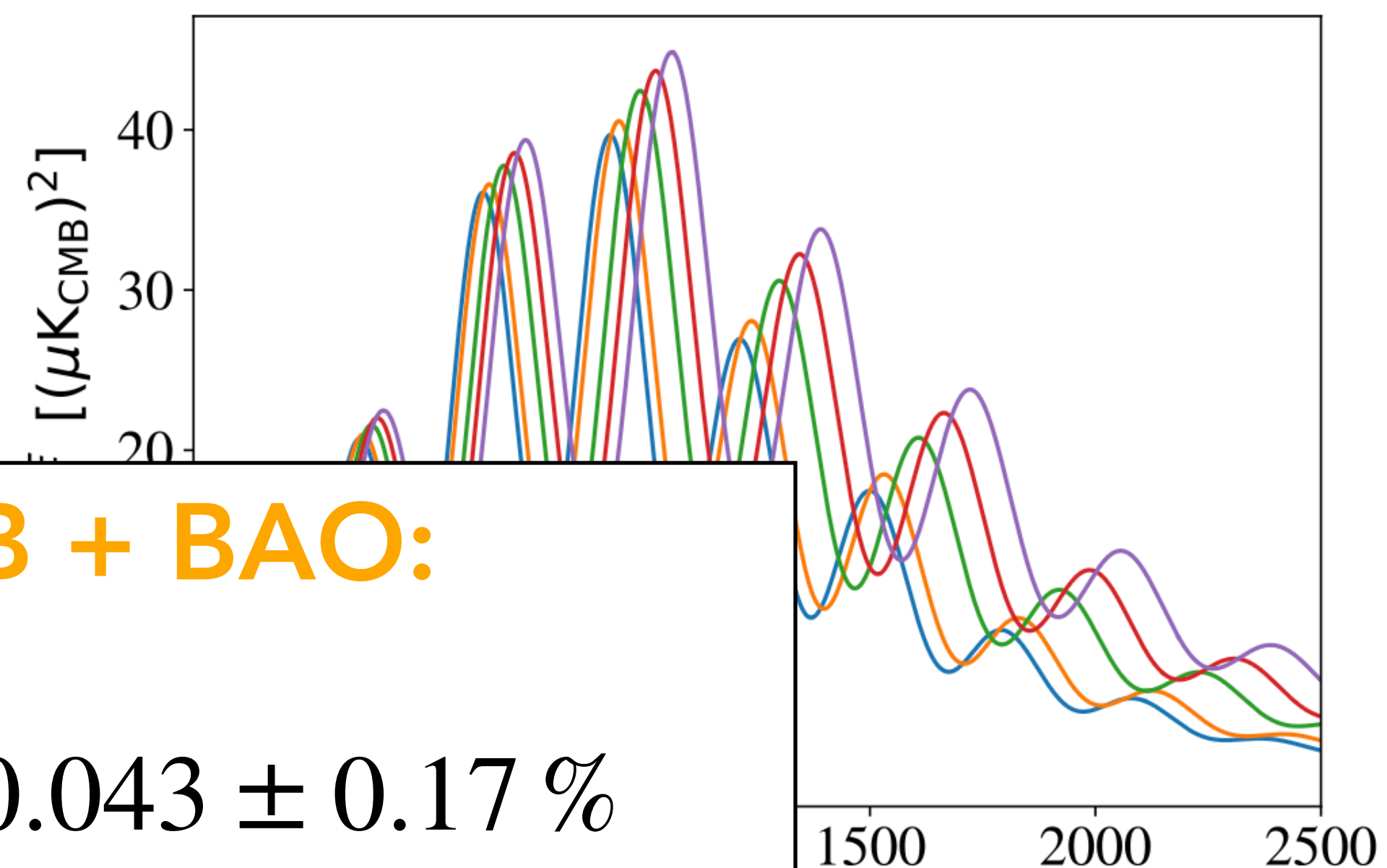
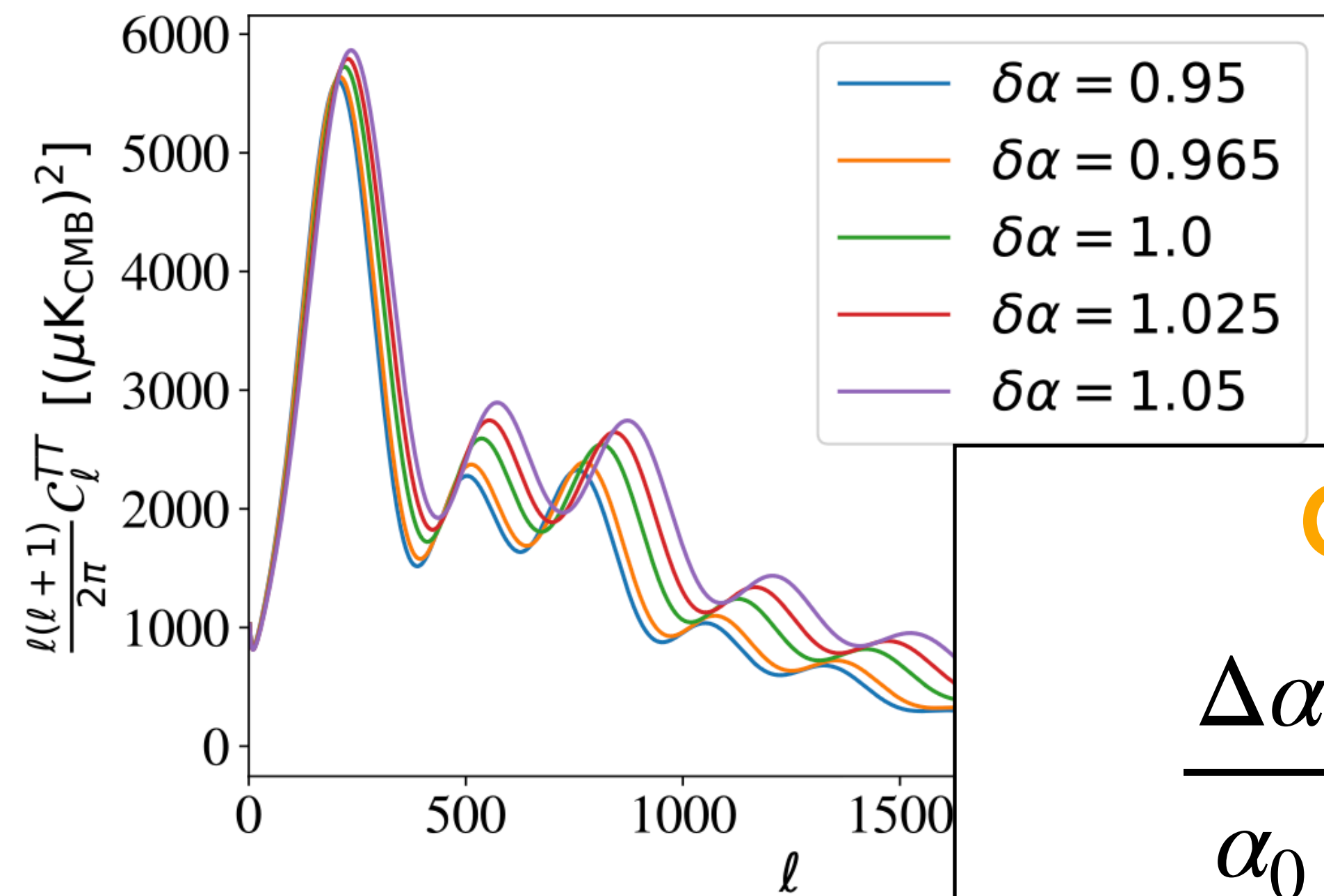
$$\frac{\Delta\alpha}{\alpha_0} = 0.037 \pm 0.17 \%$$

$$\frac{\Delta\mu}{\mu_0} = 14.4^{+6}_{-7.3} \%$$

ACT collaboration 2025







**CMB + BAO:**

$$\frac{\Delta\alpha}{\alpha_0} = 0.043 \pm 0.17 \%$$

$$\frac{\Delta\mu}{\mu_0} = 0.96 \pm 0.6 \%$$

ACT collaboration 2025



# Tests of EEP: local data are extremely sharp!

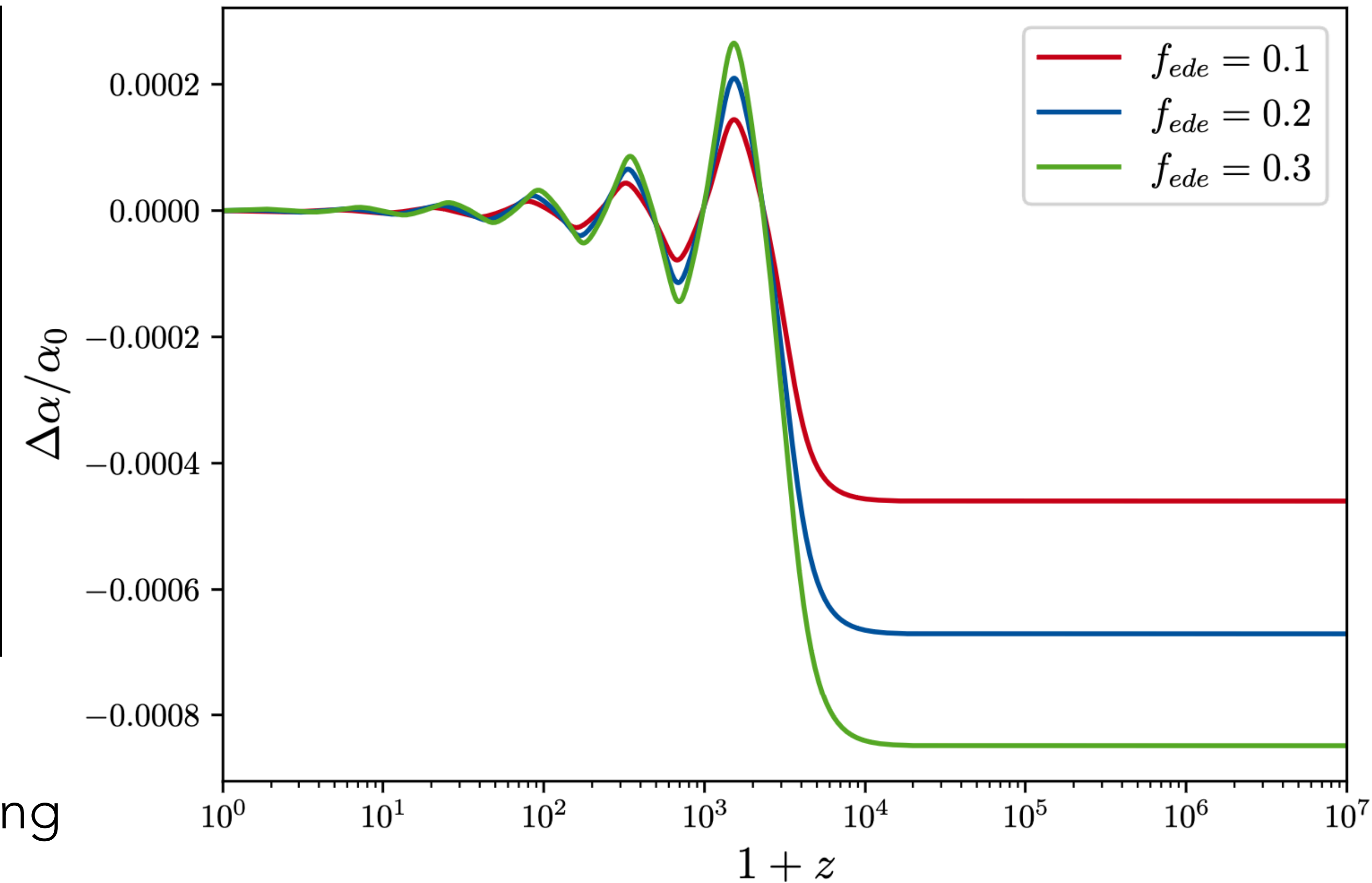
## Atomic clocks:

$$\frac{\dot{\alpha}}{\alpha_0} = (1.8 \pm 2.5) \times 10^{-19} \text{ yr}^{-1}$$

$$\frac{\dot{\mu}}{\mu_0} = (3.09 \pm 1.42) \times 10^{-17} \text{ yr}^{-1}$$

[\*Filzinger+ 2023\*](#)

It becomes thus extremely hard (and even impossible) to build simple **physical models** varying the constant at CMB to solve the  $H_0$  tension and avoiding these data without **screening**



[\*\(Vacher & Schöneberg 2024\)\*](#)



**Now, what is GR?**

**One possible metric theory of gravity!**



# A metric theory of gravity: General relativity

- (S1): Space-time is a **smooth four dimensional manifold**  $M$ .
- (S2):  $M$  is equipped with a **Lorentzian metric**  $g$  at each point.  **$g$  is the unique dynamical quantity of the theory** associated with gravitation, which space-time evolution must be found.

(These two axioms are motivated by EEP as we saw before)



# A metric theory of gravity: General relativity

- (S1): Space-time is a **smooth four dimensional manifold**  $M$ . **Higher dimension theories**
- (S2):  $M$  is equipped with a **Lorentzian metric**  $g$  at each point.  **$g$  is the unique dynamical quantity of the theory** associated with gravitation, which space-time evolution must be found.

**Additional fields of gravity**

(These two axioms are motivated by EEP as we saw before)

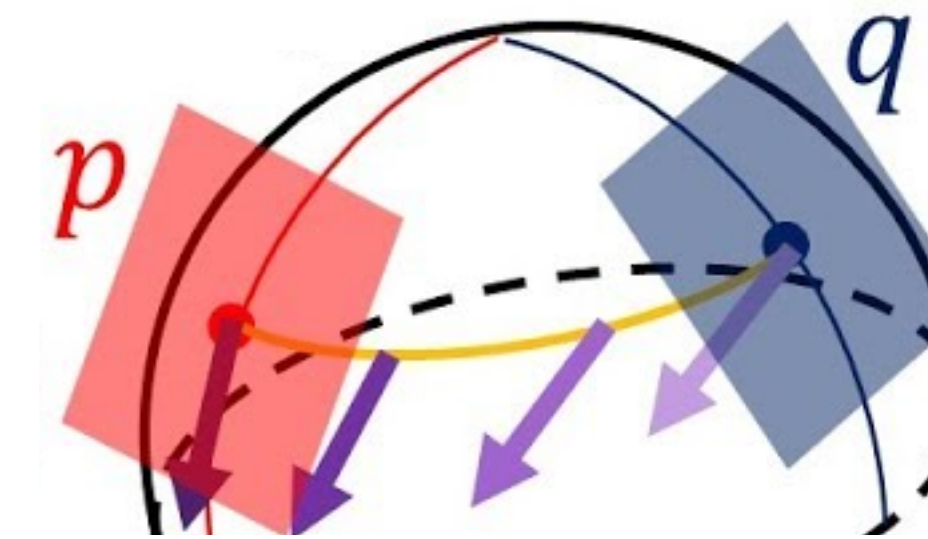


# A metric theory of gravity: General relativity

- (S3):  $M$  is equipped with a **connection**  $\nabla$  defining **parallel transport** of tangent vectors.  $\nabla$  is **metric compatible** (S3a) and **torsion free** (S3b). Hence, the unique **Levi-Civita connection**, expressed from  $g$ . **All geometrical objects on space-time** are transported through generalizations of  $\nabla$ .

In a local chart:

$$\nabla_{\mu}(v^{\lambda}e_{\lambda}) = (\partial_{\mu}v^{\lambda} + \Gamma^{\lambda}_{\mu\nu}v^{\nu})e_{\lambda}$$



$$\underbrace{\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}}_{\text{torsion-freeness}} + \underbrace{\nabla_{\rho}g_{\mu\nu} = 0}_{\text{Metric compatible}} \rightarrow \underbrace{\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma} \left( \partial_{\mu}g_{\sigma\nu} + \partial_{\nu}g_{\sigma\mu} - \partial_{\sigma}g_{\mu\nu} \right)}_{\text{Levi-Civita connection}}$$



# A metric theory of gravity: General relativity

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- (S4): Space-time contains **matter**. This matter is described by **fields**, the matter fields  $\psi$  on curved space-time. Collectively, their local **energy density, momentum density, and stress** at every point of space-time, are encoded in a symmetric (0,2)-tensor  $T_{\mu\nu}$ , called the **stress-energy tensor**.



# A metric theory of gravity: General relativity

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## Theories with torsion, non-metricity or non minimal coupling

- (S4): Space-time contains **matter**. This matter is described by **fields**, the matter fields  $\psi$  on curved space-time. Collectively, their local **energy density, momentum density, and stress** at every point of space-time, are encoded in a symmetric (0,2)-tensor  $T_{\mu\nu}$ , called the **stress-energy tensor**.

## Theories with other types of matter: arguably not modified gravity (?)



# A metric theory of gravity: General relativity

- (D1): **Einstein equations** are satisfied:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} - \Lambda g_{\mu\nu}$$

- (D2): Matter fields obey their special-relativistic field equations in any local inertial free-falling frame. A simple and general way to satisfy this demand is to leave the special-relativistic expressions untouched in other general frames with the replacement  $\eta \rightarrow g$  and  $\partial \rightarrow \nabla$ . This prescription, is sometimes known as **minimal coupling** or **universal coupling**



# A metric theory of gravity: General relativity

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Other more complicated equations  
(higher order terms)

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Lorentz violating theories  
(Cosmic birefringence etc)



# A metric theory of gravity: General relativity

- (D1 + D2): Can be contained within the single **Einstein-Hilbert action**:

$$S_{EH} = \int \sqrt{-|g|} \left( \frac{c^4}{16\pi G} (R - 2\Lambda) + \mathcal{L}_m(\psi) \right) d^4x$$

with the **stress energy tensor**:

$$T_{\mu\nu} = - \frac{2}{\sqrt{-|g|}} \frac{\delta \sqrt{-|g|} \mathcal{L}_m}{\delta g^{\mu\nu}}$$



# A metric theory of gravity: General relativity

- From D1+D2 it follows:

$$\nabla_{\mu} T^{\mu\nu} = 0$$

**Continuity equation** (good for consistency!)

$$u^{\mu} \nabla_{\mu} u_{\nu} = 0$$

**Geodesic equation** (good for EEP!)

$$\Phi = -\frac{Gm}{r}$$

**Weak field Newtonian limit** (good as its one requirement!)

see proofs in notes



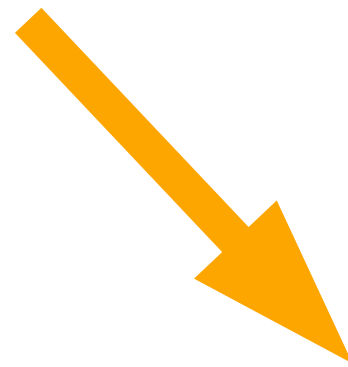
# The strong equivalence principle

Strong equivalence principle (SEP): take EEP and remove all « local non gravitational » from the text.

**Gravity also gravitates universally!**

$$S = S_g + S_m$$

EEP



$$S_m = \int \sqrt{-|g|} \mathcal{L}_m(\psi, g_{\mu\nu}) d^4x$$

$$S_g = \int \sqrt{-|g^*|} \frac{c^4}{16\pi G} R(g^*) d^4x$$

Strong equivalence principle (SEP):

$$g = g^*$$



# The strong equivalence principle

In theories violating the SEP

$$S_m = \int \sqrt{-|g|} \mathcal{L}_m(\psi, g_{\mu\nu}) d^4x$$

$$S_g = \int \sqrt{-|g^*|} \mathcal{L}_g(g^*, \phi \dots) d^4x$$

- $g = f(g^*, \phi, \dots)$ , hence a theory can be metric (EEP true) and violate the SEP (this will be the case of the great majority of theories of modified gravity we will explore)
- If Newton constant  $G$  varies with space/time the **SEP is violated!!!!** Hence  $G(x^\mu) \Rightarrow g \neq g^*$  (see the painful proof is in the notes). This is also the case in most of the modified gravity theories.



# The strong equivalence principle

$G$  is the most **poorly measured** fundamental constant (and people does not even agree on its value, the so-called « **big  $G$  problem** » [\(Gibney \(2026\)\)](#))

However, using Lunar Laser Ranging (LLR):

$$\left| \frac{\dot{G}}{G} \right| = (-5.0 \pm 9.6) \times 10^{-15} \text{ yr}^{-1}$$

But poorly measured with cosmology,  $\sim 10\%$  with CMB



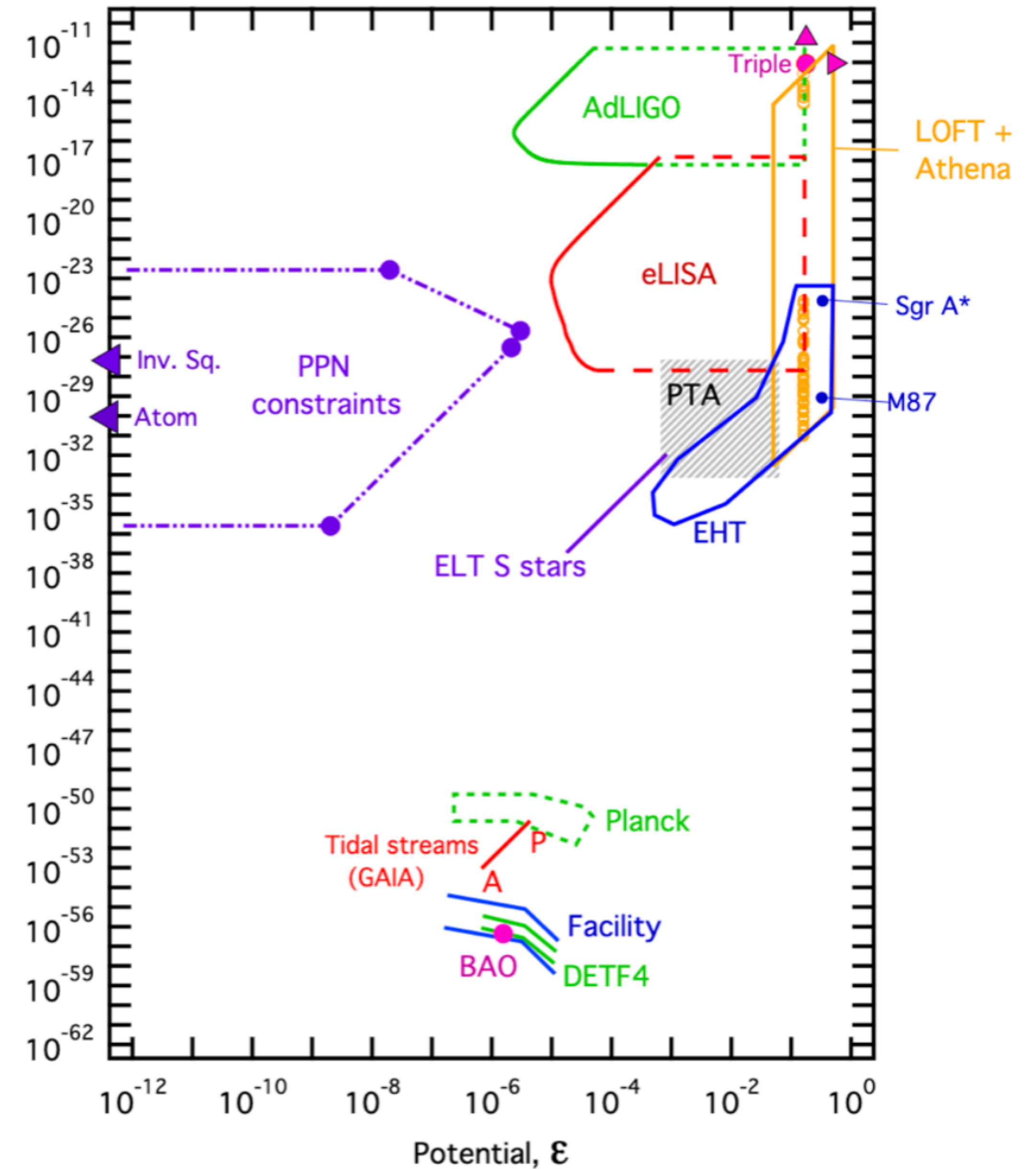
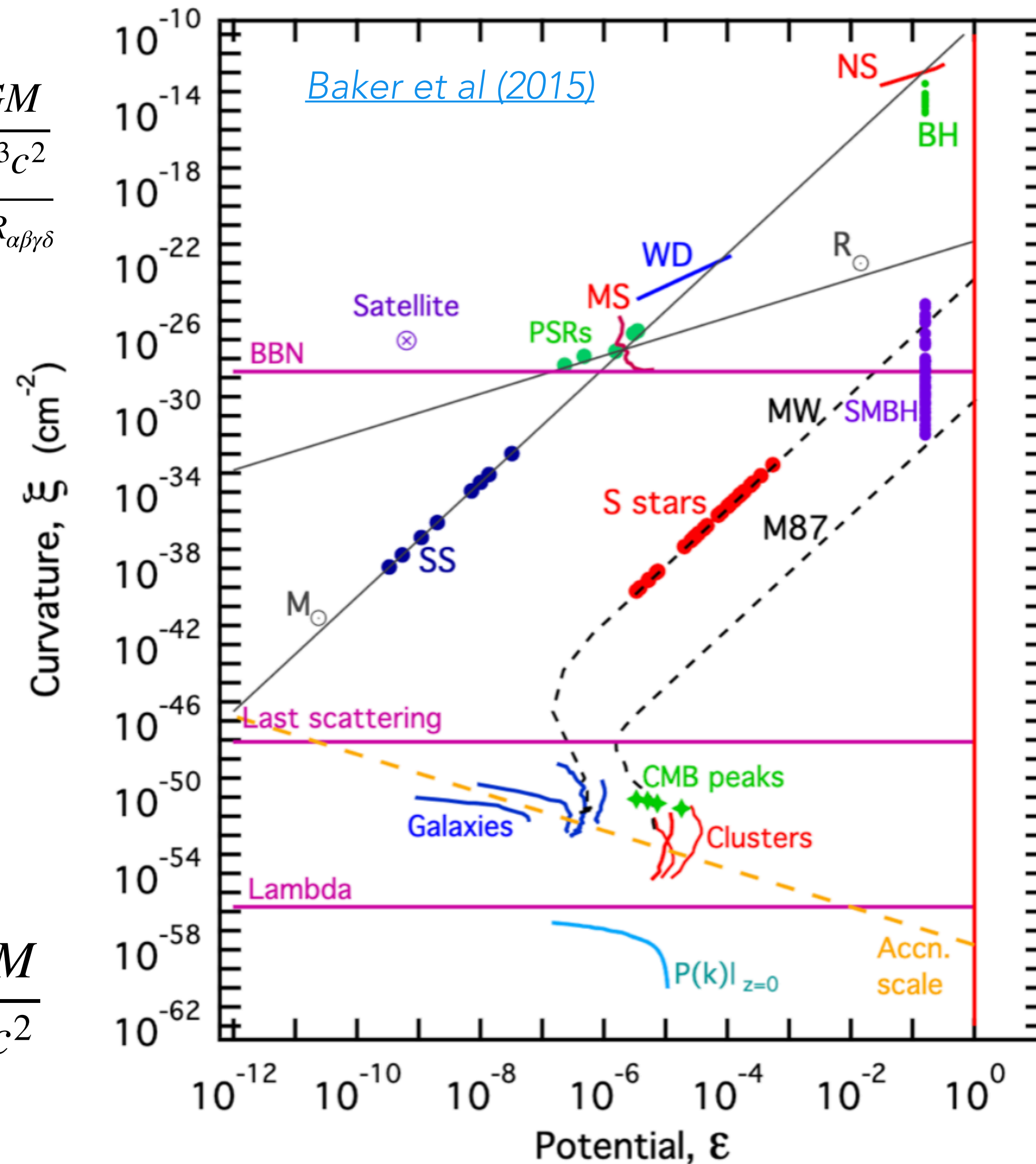


# Testing GR on every scales (see notes for all constraints)

$$\xi = \sqrt{48} \frac{GM}{R^3 c^2}$$

$$= \sqrt{R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}}$$

$$\varepsilon = \frac{GM}{Rc^2}$$

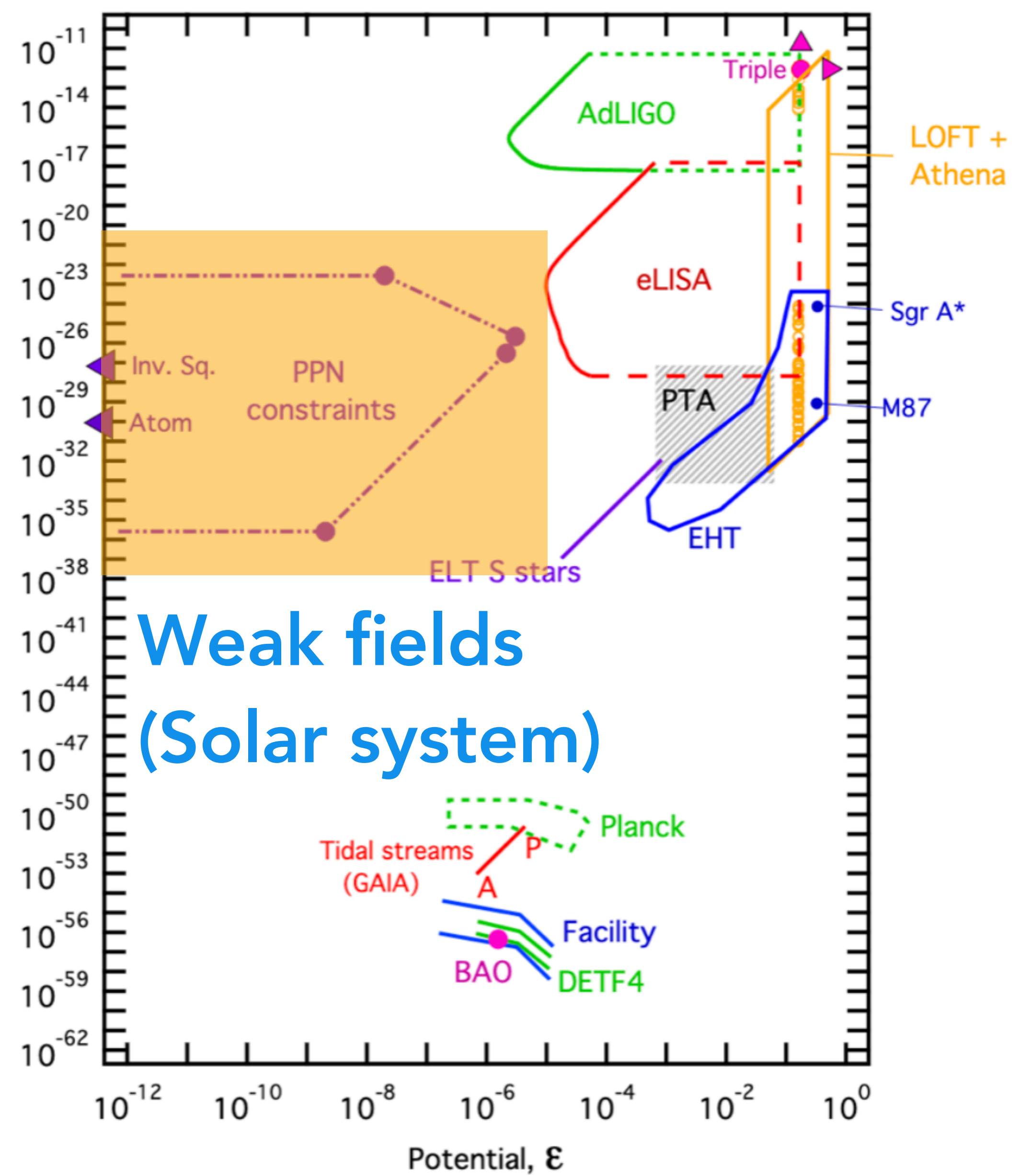
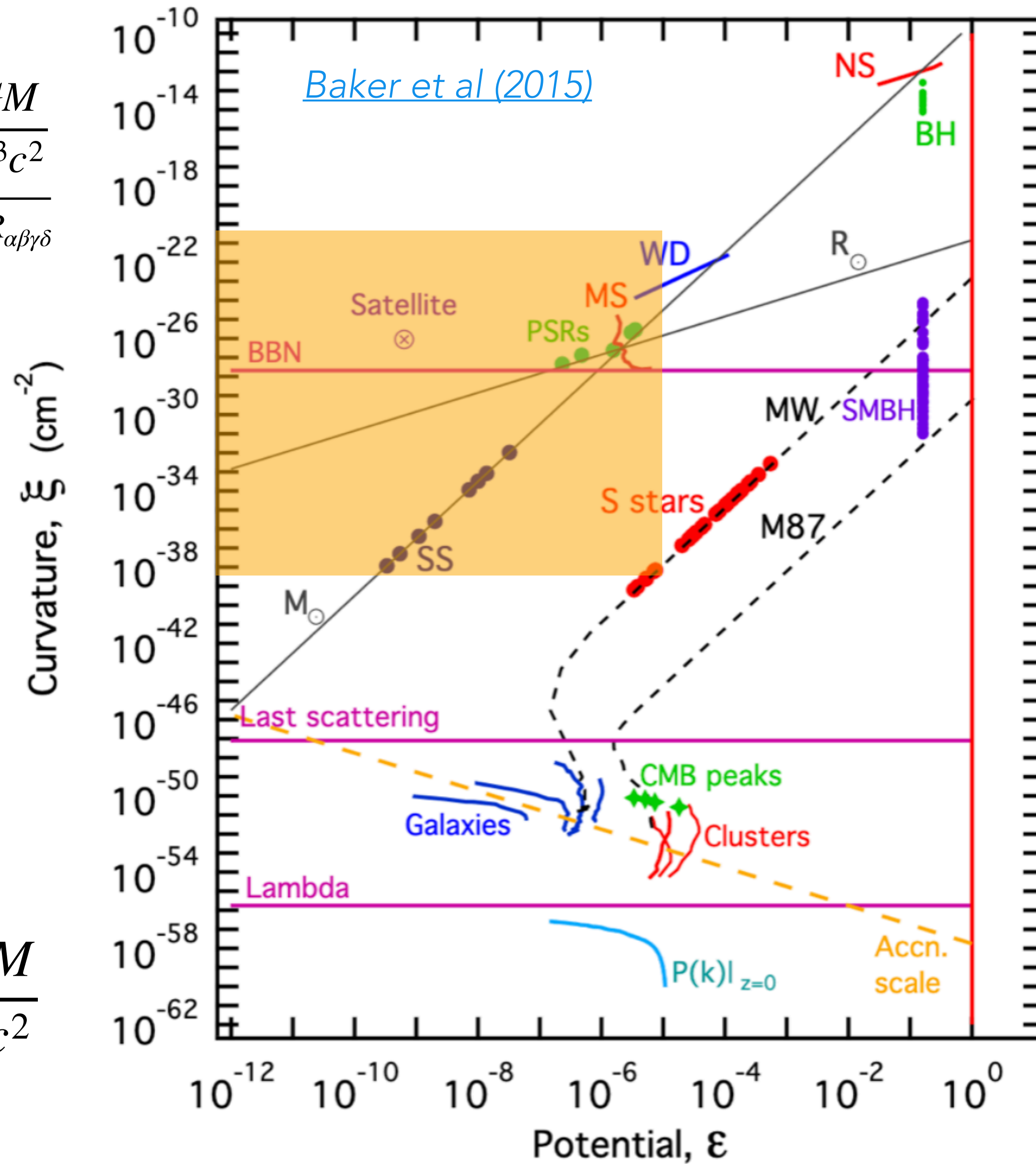




# Testing GR on every scales (see notes for all constraints)

$$\begin{aligned}\xi &= \sqrt{48} \frac{GM}{R^3 c^2} \\ &= \sqrt{R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}}\end{aligned}$$

$$\varepsilon = \frac{GM}{Rc^2}$$





# Testing GR on every scales (see notes for all constraints)

PPN parameters:

$$g_{00} \simeq -1 + 2\frac{GM}{c^2r} - 2\beta\left(\frac{GM}{c^2r}\right)^2$$

$$g_{jk} \simeq \left(1 + 2\gamma\frac{GM}{c^2r}\right)\delta_{jk}$$

| Parameter      | Best constraint                     | Effect                                                           | Source (year)                                                                                                                              |
|----------------|-------------------------------------|------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------|
| $\gamma - 1$   | $(2.1 \pm 2.3) \times 10^{-5}$      | Shapiro time delay                                               | Cassini – <a href="#">Bertotti et al. (2003)</a>                                                                                           |
| $\beta - 1$    | $(-1.6 \pm 1.8) \times 10^{-5}$     | Mercury perihelion + solar $J_2$                                 | MESSENGER – <a href="#">Genova et al. (2018)</a>                                                                                           |
| $\xi$          | $3.9 \times 10^{-9}$                | Spin precession                                                  | Solitary millisecond pulsars – <a href="#">Shao &amp; Wex (2013)</a>                                                                       |
| $\alpha_1$     | $-0.4^{+3.7}_{-3.1} \times 10^{-5}$ | Orbital polarization                                             | PSR J1738+0333 – <a href="#">Shao &amp; Wex (2012)</a>                                                                                     |
| $\alpha_2$     | $1.6 \times 10^{-9}$                | Spin precession                                                  | Solitary millisecond pulsars – <a href="#">Shao et al. (2013)</a>                                                                          |
| $\alpha_3$     | $4 \times 10^{-20}$                 | Self-acceleration                                                | Pulsar spin-down statistics – <a href="#">Will (2014)</a>                                                                                  |
| $\zeta_1$      | 0.02                                | Combined PPN bounds                                              | <a href="#">Will (2014)</a>                                                                                                                |
| $\zeta_2$      | $1.3 \times 10^{-5}$                | Binary-pulsar self-acceleration ( $\ddot{\nu}, \ddot{\nu}'$ )    | Combined binary pulsars – <a href="#">Miao et al. (2020)</a> , improving <a href="#">Will (1992)</a>                                       |
| $\zeta_3$      | $1 \times 10^{-8}$                  | Newton’s 3rd law (lunar acceleration)                            | <a href="#">Bartlett &amp; Van Buren (1986)</a>                                                                                            |
| $\zeta_4$      | 0.006                               | Not independent:<br>$6\zeta_4 = 3\alpha_3 + 2\zeta_1 - 3\zeta_3$ | <a href="#">Will (1976)</a>                                                                                                                |
| $\eta_N$ (SEP) | $(-6.6 \pm 7.2) \times 10^{-5}$     | Nordtvedt effect (Not independent, see below)                    | MESSENGER – <a href="#">Genova et al. (2018)</a> ; LLR gives $(-0.2 \pm 1.1) \times 10^{-4}$ – <a href="#">Hofmann &amp; Müller (2018)</a> |

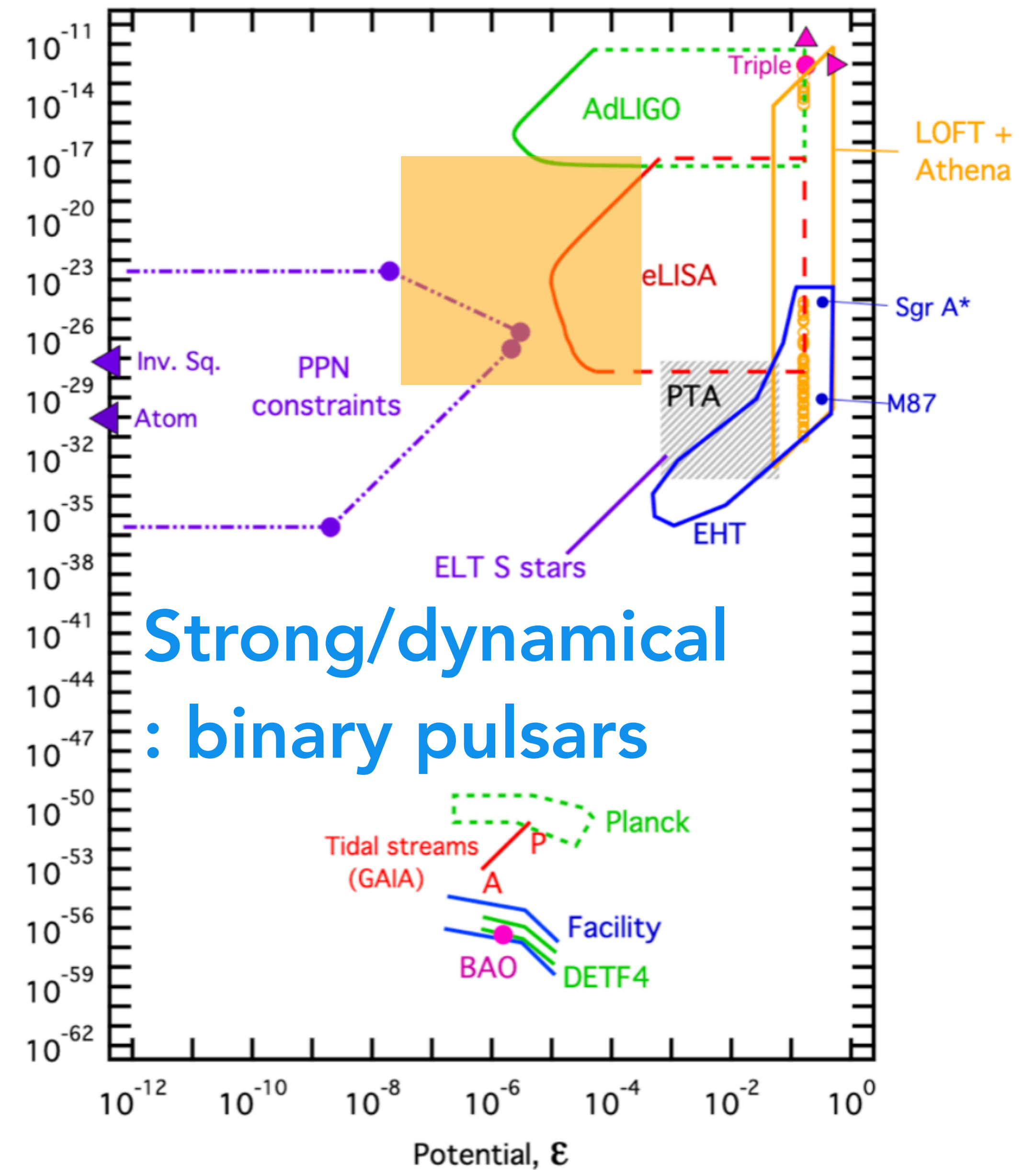
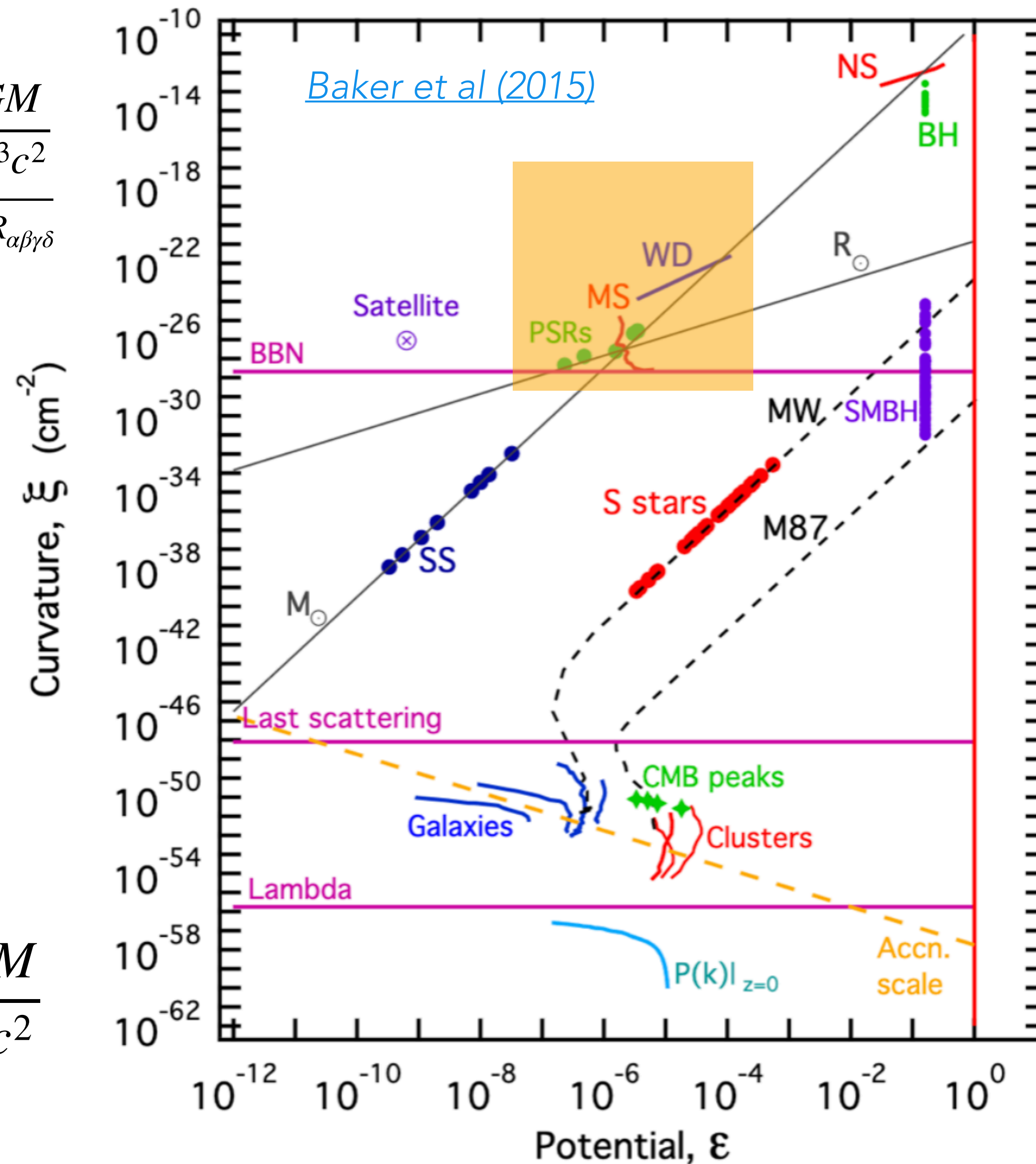


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$$= \sqrt{R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}}$$

$$\varepsilon = \frac{GM}{Rc^2}$$





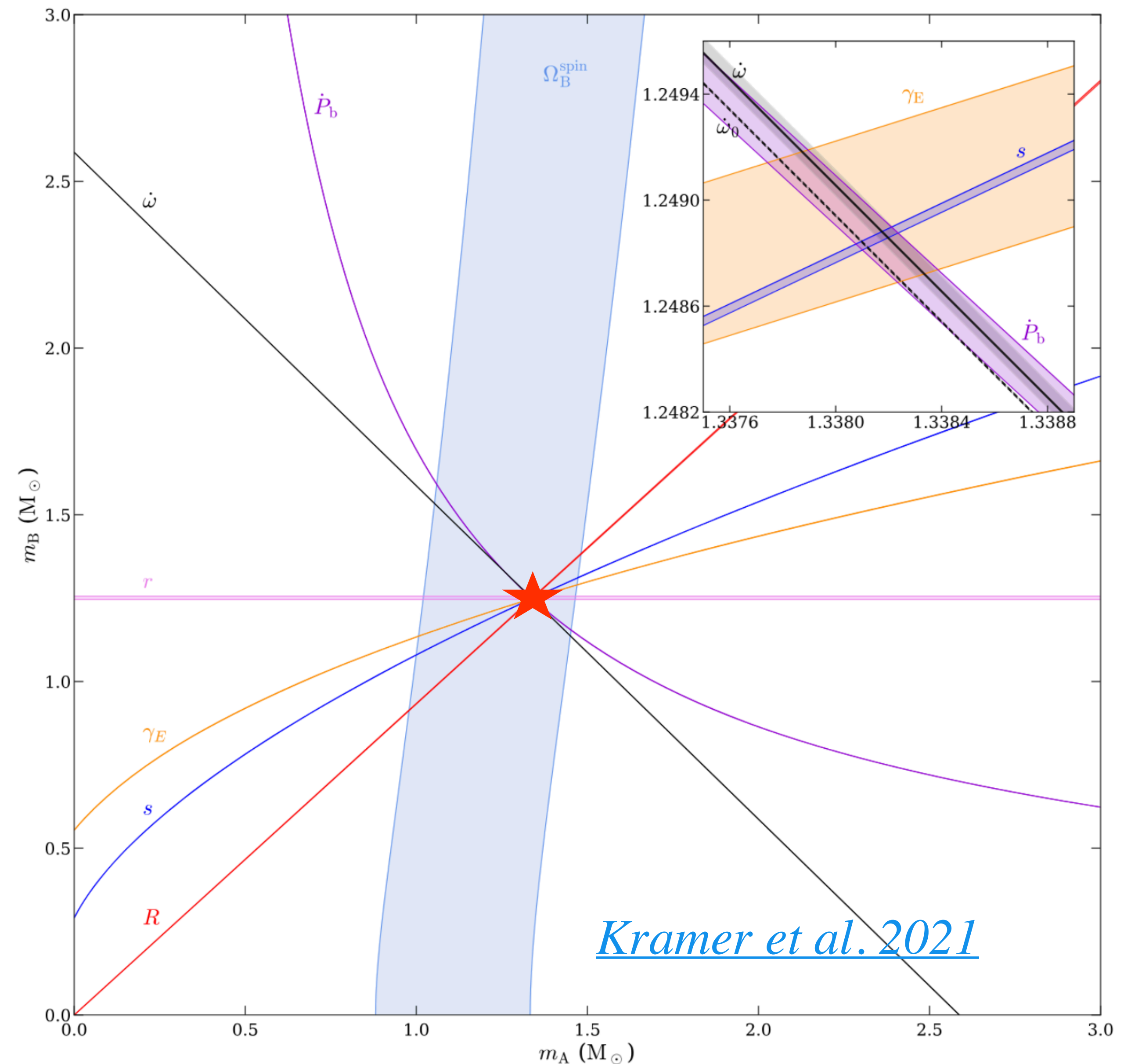
# Testing GR on every scales (see notes for all constraints)

## Binary pulsars and PKP:

$\sim 10^{-4}$  level validation of GR  
in dynamical/strong regime with  
 $\dot{P}_b$  and post-keplerian parameters (PKP)

Constraints on SEP: ([Voisin et al. 2020](#))

$$\left| 1 - \frac{m_i}{m_g} \right| \lesssim 2 \times 10^{-6}$$



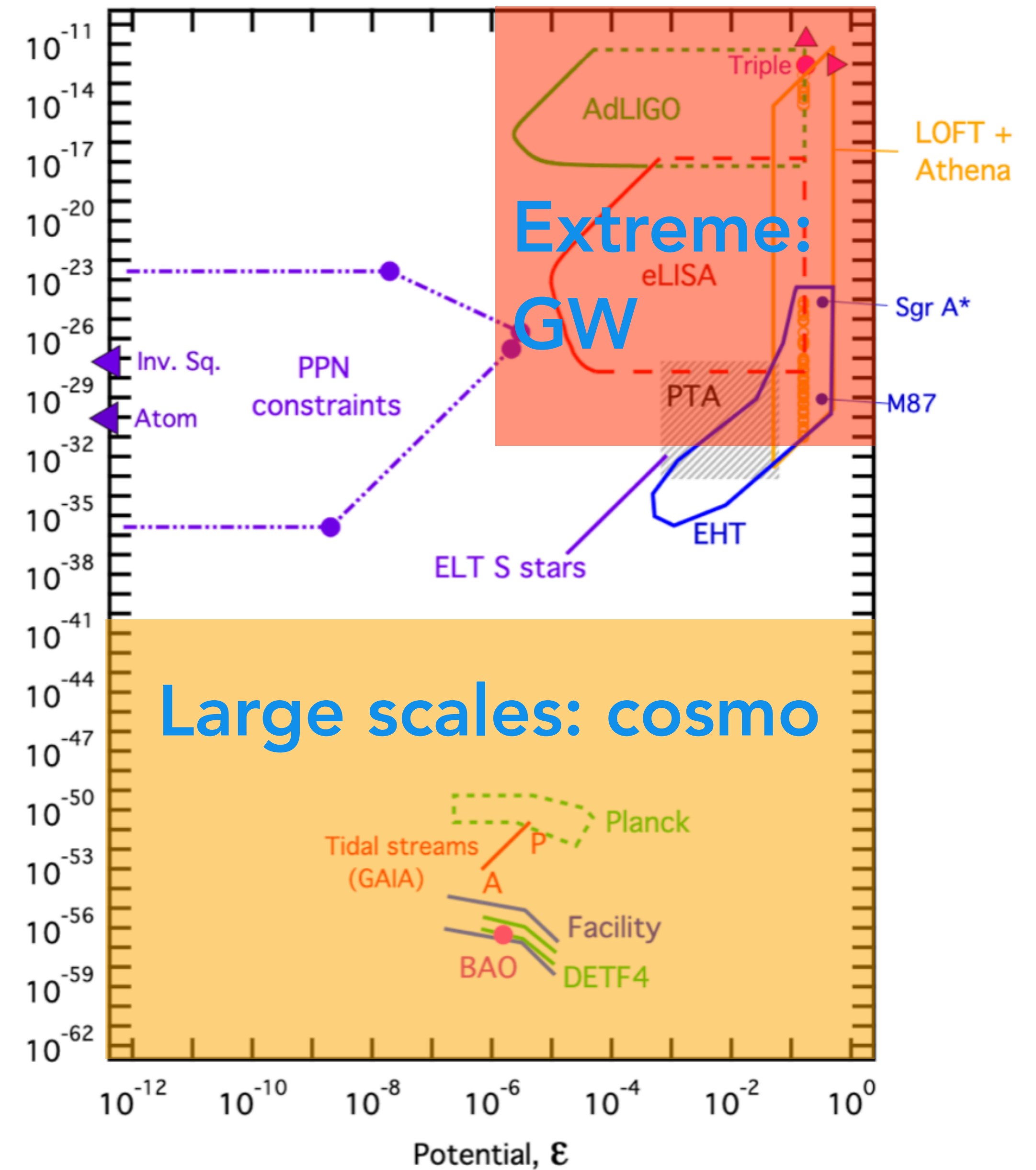
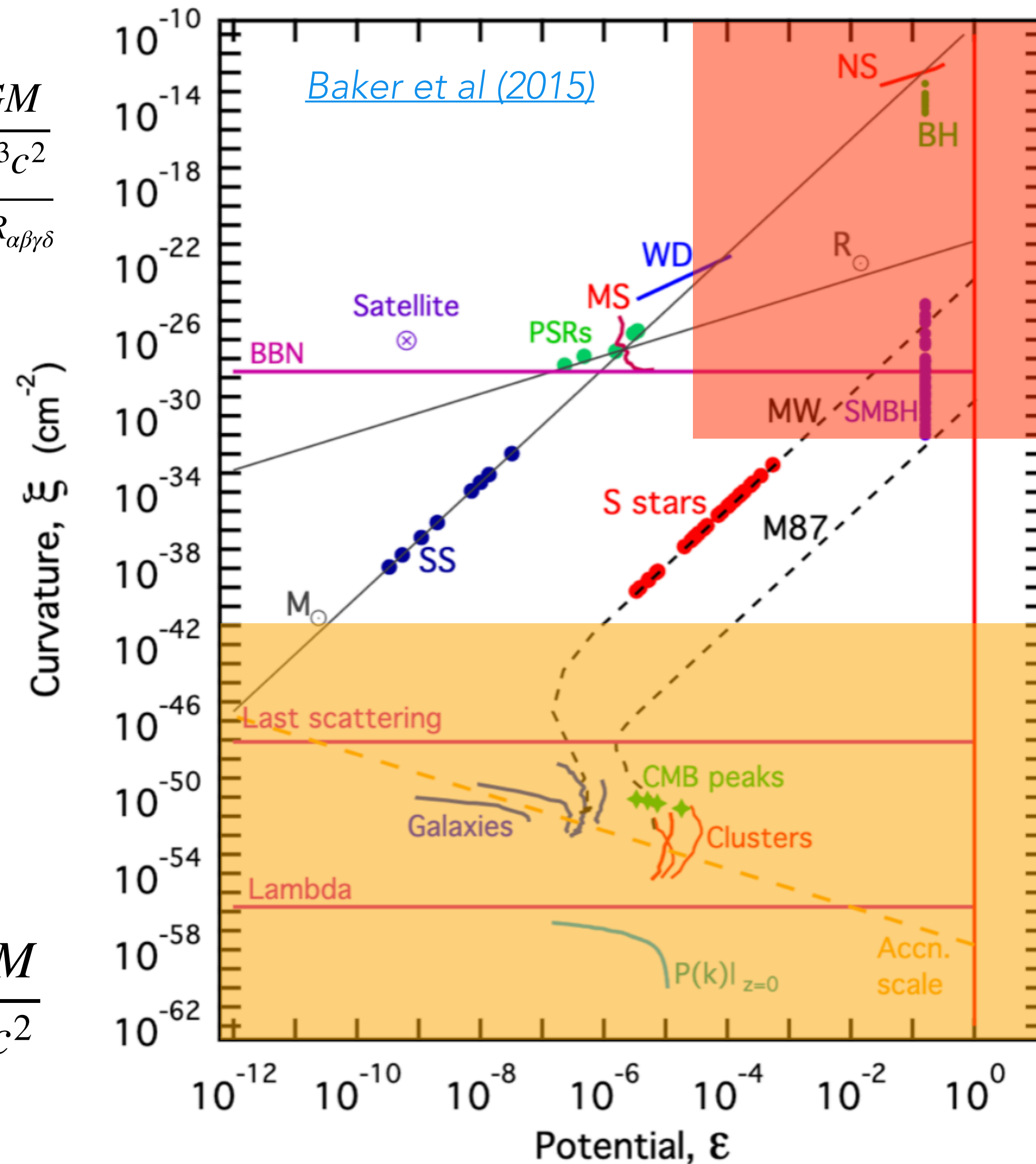


# Testing GR on every scales (see notes for all constraints)

$$\xi = \sqrt{48} \frac{GM}{R^3 c^2}$$

$$= \sqrt{R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}}$$

$$\varepsilon = \frac{GM}{Rc^2}$$





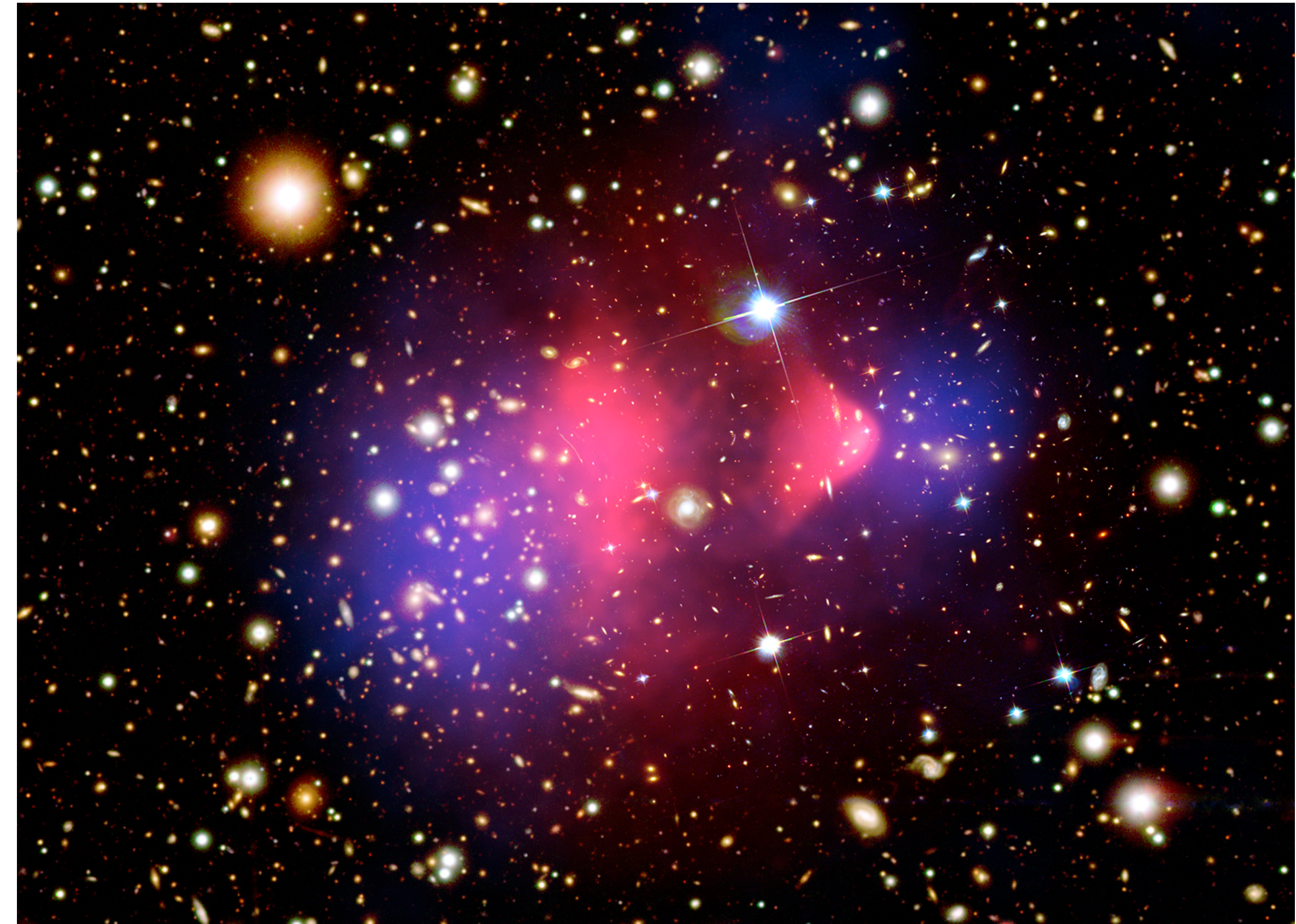
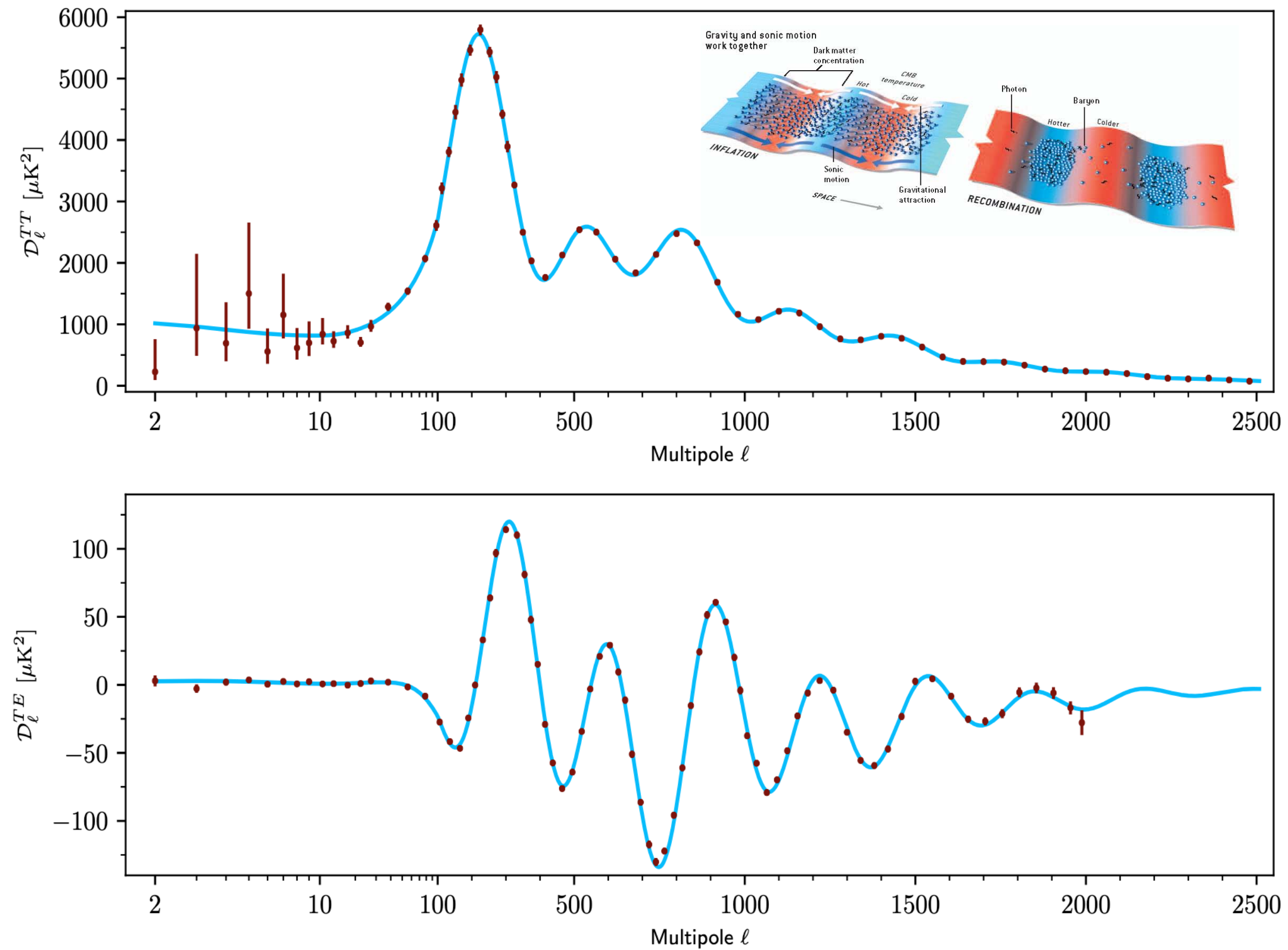
# Can modifications of GR solve cosmological puzzles?

**Dark energy:** You don't need modgrav, GR already does, with  $\Lambda$ . As you may know the big problem is the  $10^{120}$  orders of difference with QFT prediction ([Weinberg 1989](#)). Now is it a gravity or a QFT problem? Its **both** ! This is exactly at the interface  $\rightarrow$  quantum gravity  
Many modified gravity try to generate dark energy and supersede  $\Lambda$ . This will **not be our priority here but we will encounter some**. However, all of them face fine tuning, struggle to solve the CC problem (yet another Weinberg no-go theorem).

**Dark matter:** Attempts at modifying gravity to explain DM : **MOND**  
Explain very well rotation curves and predicted Tully fisher relation  
Struggle with two observables: **Bullet cluster** and **CMB acoustic peaks**. (Next slide)  
Relativistic generalisations, **TeVeS theories** solves it but at a big fine tuning price  
+ need new fields = new matter?



# Can modifications of GR solve cosmological puzzles?



Planck (2018) I



# GR and cosmology

**Question for you:** what would it change for cosmology if gravity theory was not GR,  
Let's say Newtonian theory for example?



# GR and cosmology

**Question for you:** what would it change for cosmology if gravity theory was not GR,  
Let's say Newtonian theory for example?

**Answer:** nothing at the background level! Friedmann equations can be written in Newtonian theory, even including spatial curvature as a boundary condition (see notes)

$$H^2 = \sum_i \frac{8\pi G \rho_i}{3}$$

In order to probe properly modified gravity one must study **perturbations (structure formation) and the effect on light (lensing)**



# Perturbations in a FLRW metric

First order perturbation of the FLRW metric (Newton gauge, no vectors):

$$g + \delta g \simeq - \left( 1 + \frac{2\Psi}{c^2} \right) c^2 dt \otimes dt + a^2(t) \left[ \left( 1 - \frac{2\Phi}{c^2} \right) \delta_{ij} + h_{ij} \right] dx^i \otimes dx^j$$

Lapse function ~  
Time distortions = Newtonian  
gravitational potential

Perturbation of scale factor

Tensor perturbations



# Perturbations in a FLRW metric

First order perturbation of the FLRW metric (Newton gauge, no vectors):

$$g + \delta g \simeq - \left( 1 + \frac{2\Psi}{c^2} \right) c^2 dt \otimes dt + a^2(t) \left[ \left( 1 - \frac{2\Phi}{c^2} \right) \delta_{ij} + h_{ij} \right] dx^i \otimes dx^j$$

$$(T + \delta T)^0_0 = -(\bar{\rho} + \delta\rho), \quad (T + \delta T)^0_i = (\bar{\rho} + \bar{P}) v_i, \quad (T + \delta T)^i_j = (\bar{P} + \delta P) \delta^i_j + \Pi^i_j,$$

[See J. Lesgourgues classes and notes!](#)  
[+ Baumann \(2022\) and Dodelson \(2025 3rd ed\)](#)



# Perturbations in a FLRW metric

$$g + \delta g \simeq - \left( 1 + \frac{2\Psi}{c^2} \right) c^2 dt \otimes dt + a^2(t) \left[ \left( 1 - \frac{2\Phi}{c^2} \right) \delta_{ij} + h_{ij} \right] dx^i \otimes dx^j$$

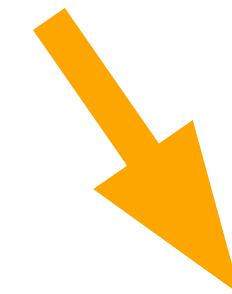


Einstein equations (subhorizon + no anisotropic stress)



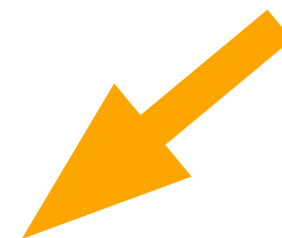
$$\Psi \simeq \Phi$$

Poisson equation



$$\ddot{h}_{ij} + 3H\dot{h}_{ij} + c^2 \frac{k^2}{a^2} h_{ij} = 0$$

Gravitational waves



$$\frac{k^2}{a^2} \Phi = -4\pi G \bar{\rho} \Delta$$

Poisson equation



# Perturbations in a FLRW metric

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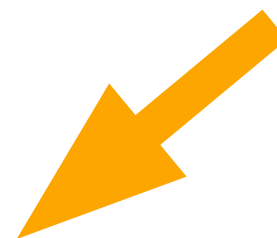
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$$\frac{k^2}{a^2} \Phi = -4\pi G \bar{\rho} \Delta$$

Poisson equation

$$\Delta \equiv \frac{\delta \rho}{\bar{\rho}} + 3 \frac{aH}{k^2} (1 + w) \partial_i v^i$$

# Perturbations in a FLRW metric

$$g + \delta g \simeq - \left( 1 + \frac{2\Psi}{c^2} \right) c^2 dt \otimes dt + a^2(t) \left[ \left( 1 - \frac{2\Phi}{c^2} \right) \delta_{ij} + h_{ij} \right] dx^i \otimes dx^j$$

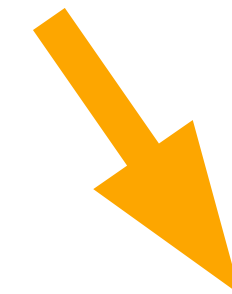


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# Perturbations in a FLRW metric: tensor modes

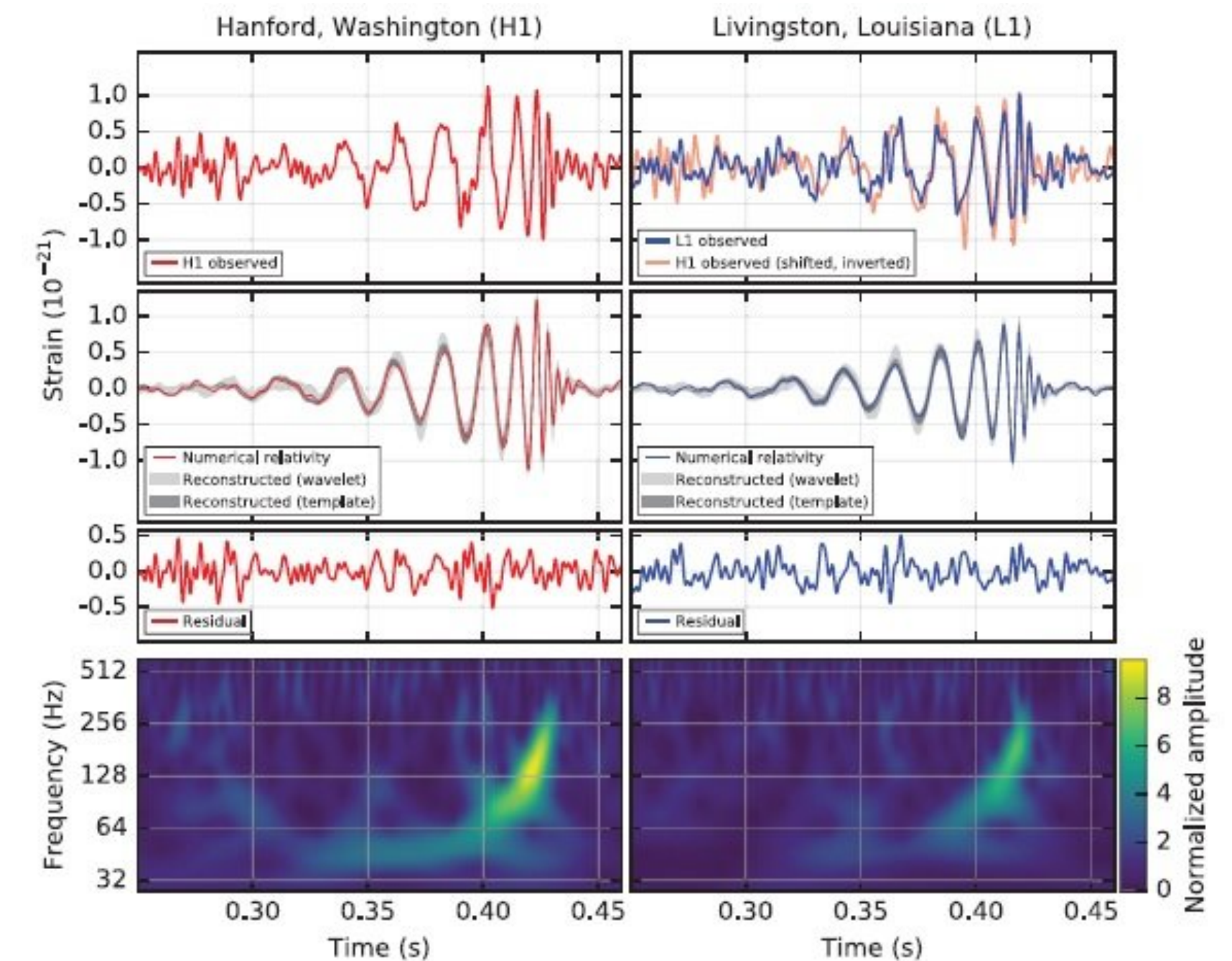
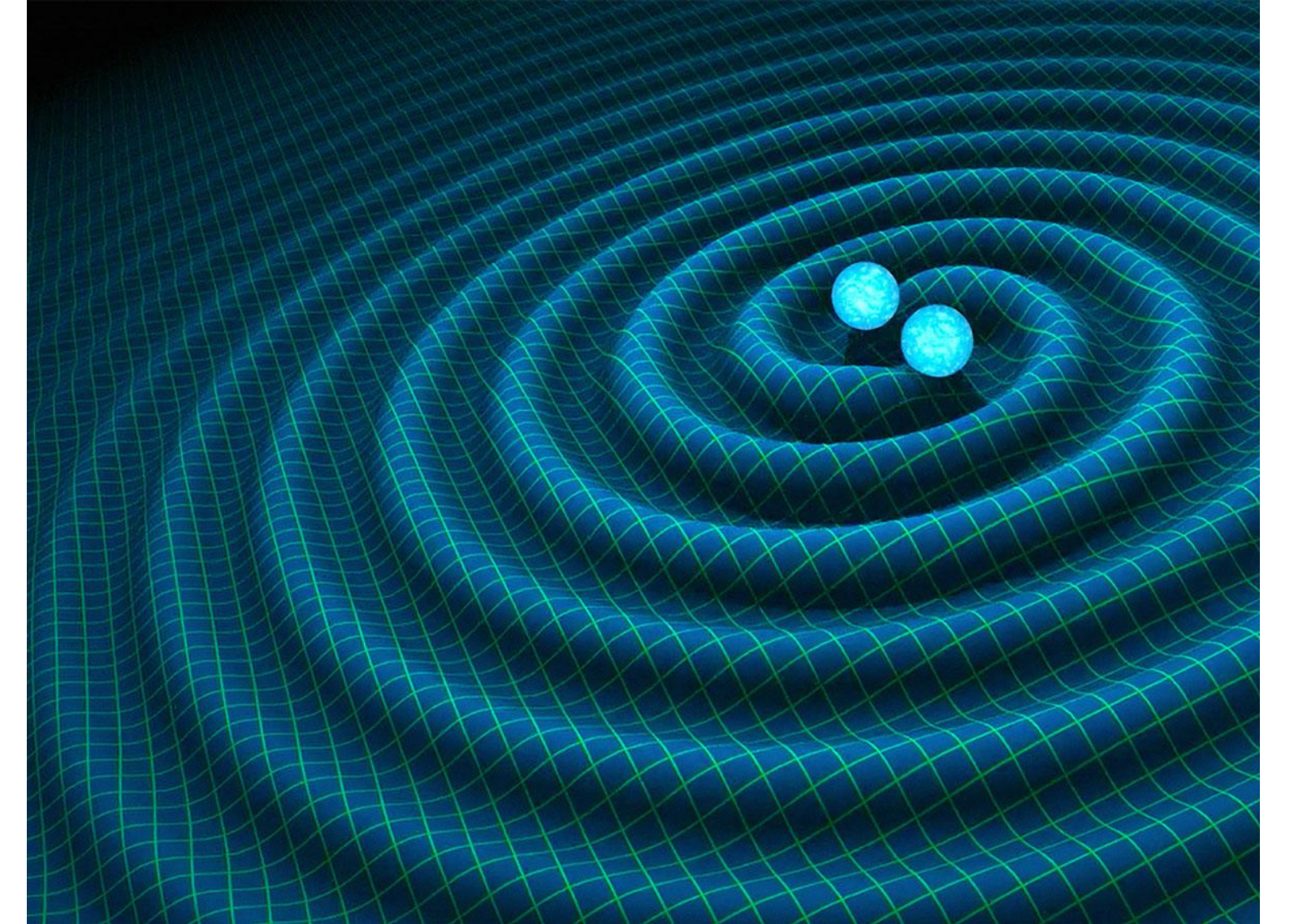
The tensor part describes the **propagation of gravitational waves**

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} + c^2 \frac{k^2}{a^2} h_{ij} = 0$$

This is a standard wave equation of velocity  $c$  in Fourier space:

$$-\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{with} \quad \partial_x \rightarrow ik, \partial_x^2 \rightarrow -k^2$$

And with an additional Hubble friction term due to expansion





# Perturbations in a FLRW metric: tensor modes

It can be extended in full generality based on standard wave physics:

$$\ddot{h}_{ij} + 3H(1 + \nu_T)\dot{h}_{ij} + (c_T^2 \frac{k^2}{a^2} + M_g^2)h_{ij} = 0$$

**Damping**

(Change of amplitude in propagation)

**Velocity  
of propagation**

**Dispersion/mass**

$$\omega^2 = c_T^2 k^2 + m^2$$

With  $m \neq 0$  different  
Modes  $k$  propagate with  
different (group) velocities  
 $d\omega/dk$

**Any theory of gravity** predict specific values for  $\nu_T, c_T, m_g$ . In GR  $\nu_T = 0, c_T = c, M_g = 0$



# Perturbations in a FLRW metric: tensor modes

$$\ddot{h}_{ij} + 3H(1 + \nu_T)\dot{h}_{ij} + (c_T^2 \frac{k^2}{a^2} + M_g^2)h_{ij} = 0$$

Velocity  
of propagation

$$-3 \times 10^{-15} < \frac{c_{\text{gw}}}{c} - 1 < 7 \times 10^{-16}$$

[Abbott et al \(2017\)](#)

Dispersion/mass

$$M_g \leq 1.27 \times 10^{-23} \text{ eV}/c^2$$

[GWTC-3 \(2025\)](#)

Damping

Poorly constrained!

$$\Xi_0 = 1.2^{+0.8}_{-0.4}$$
$$d_{\text{GW}}/d_L$$

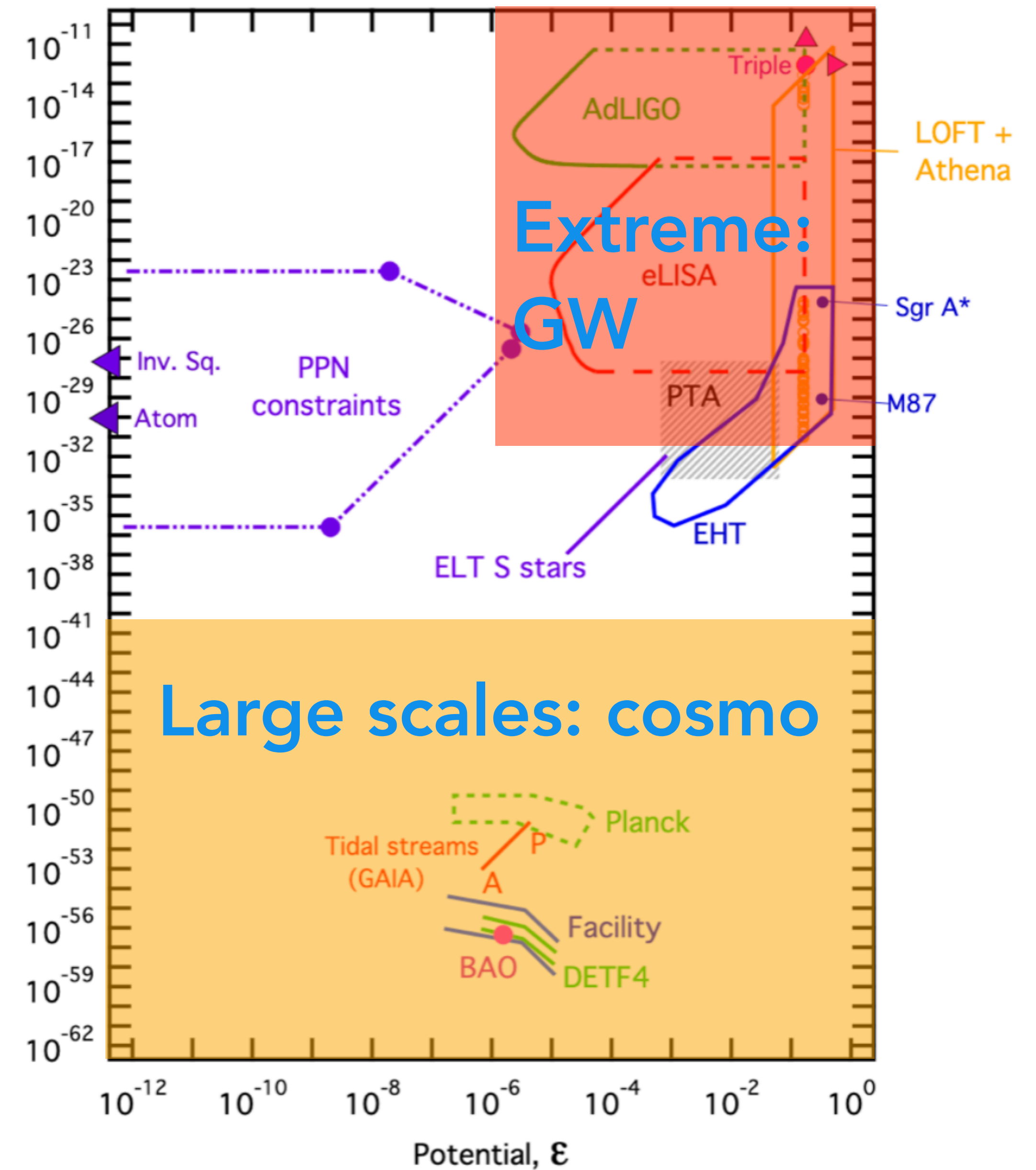
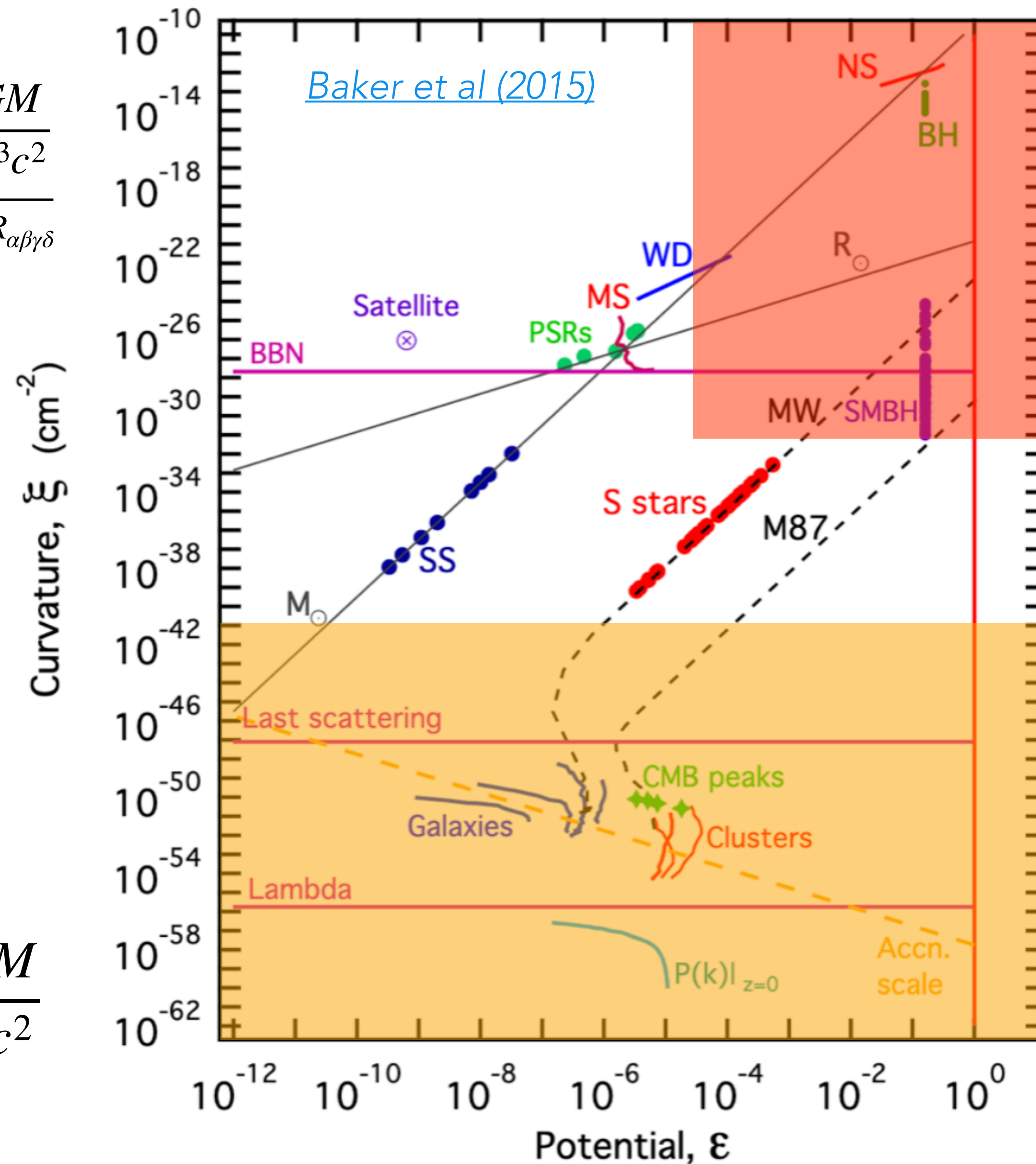
[Belgacem \(2018\)](#)

# Testing GR on every scales (see notes for all constraints)

$$\xi = \sqrt{48} \frac{GM}{R^3 c^2}$$

$$= \sqrt{R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}}$$

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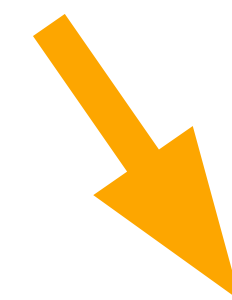


# Perturbations in a FLRW metric

$$g + \delta g \simeq - \left( 1 + \frac{2\Psi}{c^2} \right) c^2 dt \otimes dt + a^2(t) \left[ \left( 1 - \frac{2\Phi}{c^2} \right) \delta_{ij} + h_{ij} \right] dx^i \otimes dx^j$$



Einstein equations (subhorizon + no anisotropic stress)



$$\frac{k^2}{a^2} \Phi = -4\pi G \bar{\rho} \Delta$$

Poisson equation

$$\Delta \equiv \frac{\delta \rho}{\bar{\rho}} + 3 \frac{aH}{k^2} (1 + w) \partial_i v^i$$

$$\Psi \simeq \Phi$$

Poisson equation

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} + c^2 \frac{k^2}{a^2} h_{ij} = 0$$

Gravitational waves

# Perturbations in a FLRW metric: scalar modes

$$\frac{k^2}{a^2} \Phi = -4\pi G \bar{\rho} \Delta$$

$$\Psi \simeq \Phi$$



$$\frac{k^2}{a^2} \Phi = -4\pi G \mu(z, k) \bar{\rho} \Delta$$

Effective  $G$

Growth of structure

$$\frac{k^2}{a^2} \left( \frac{\Phi + \Psi}{2} \right) = -4\pi G \Sigma(z, k) \bar{\rho} \Delta$$

Weyl potential

Lensing

$$\frac{\Phi}{\Psi} = \eta(z, k)$$

Gravitational slip

Lensing

Each **modified theory of gravity** has its **own prediction** for the functions  $\mu(z, k)$ ,  $\Sigma(z, k)$  and  $\eta(z, k)$



# Perturbations in a FLRW metric: scalar modes

$$\frac{k^2}{a^2} \Phi = -4\pi G \bar{\rho} \Delta$$

$$\Psi \simeq \Phi$$



$$\frac{k^2}{a^2} \Phi = -4\pi G \mu(z, k) \bar{\rho} \Delta$$

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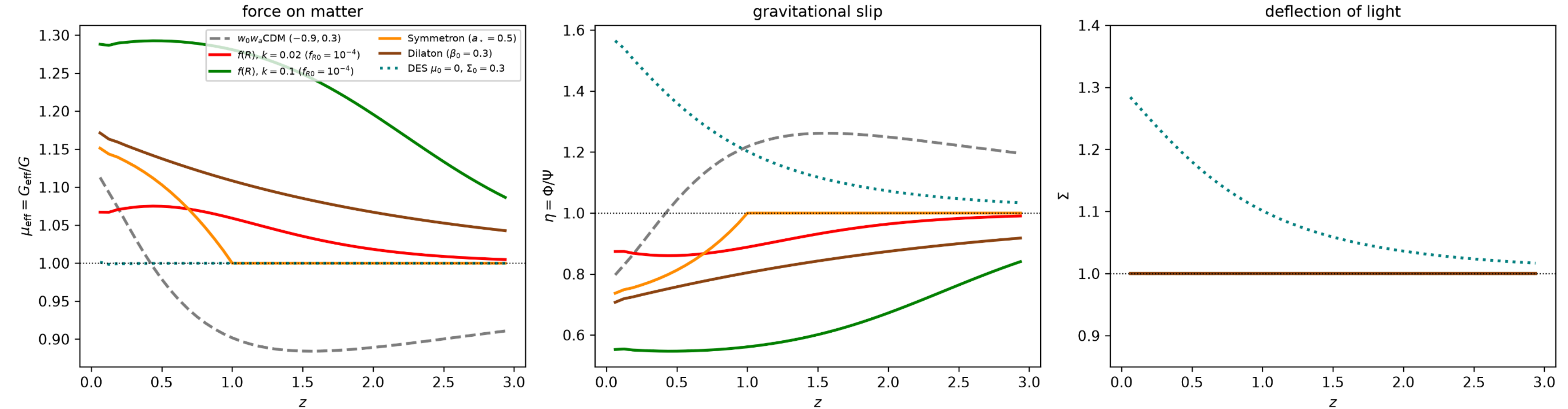
Gravitational slip

Lensing

In the next slides, all the plots I made myself are using: <https://github.com/sfu-cosmo/MGCAMB>

All tutorials are available with the notes

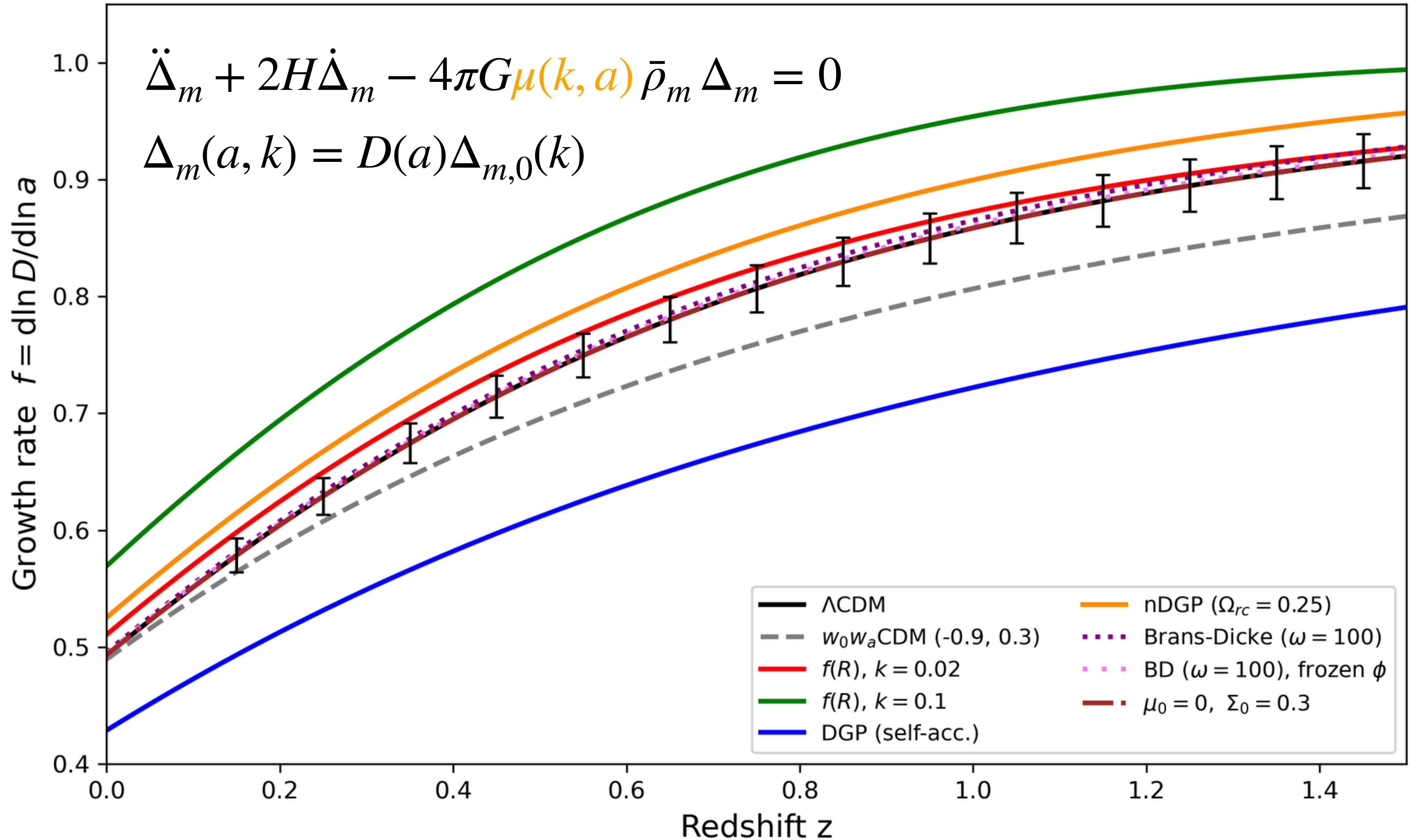
# Perturbations in a FLRW metric: scalar modes



Most models considered today are conformal,  
 their effect can be understood as  $g_{\mu\nu} \rightarrow A^2 g_{\mu\nu}$ , they do not impact light  $\Sigma = 1$



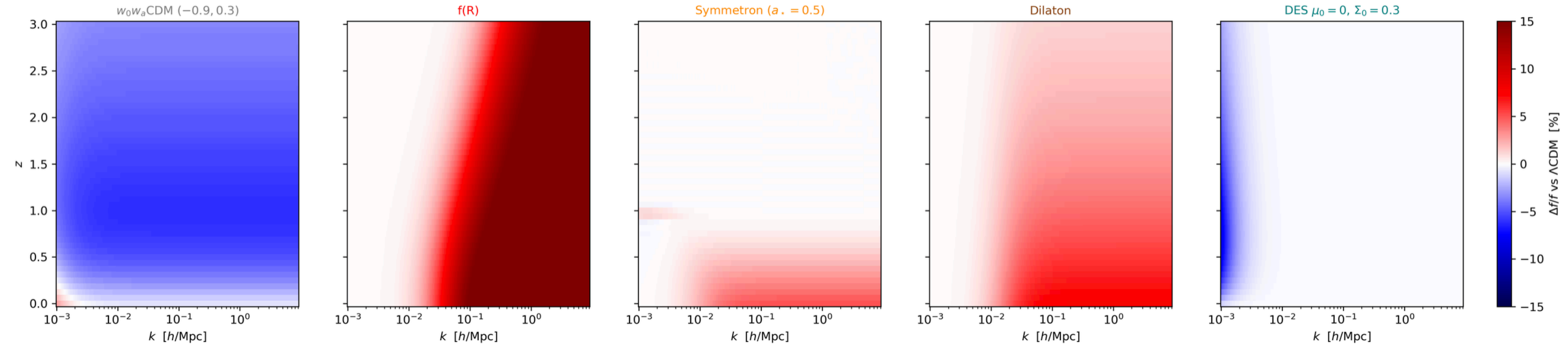
Growth rate across the modified-gravity zoo (mock 2.5% errors on  $\Lambda$ CDM)



# $\mu$ and the growth of structures

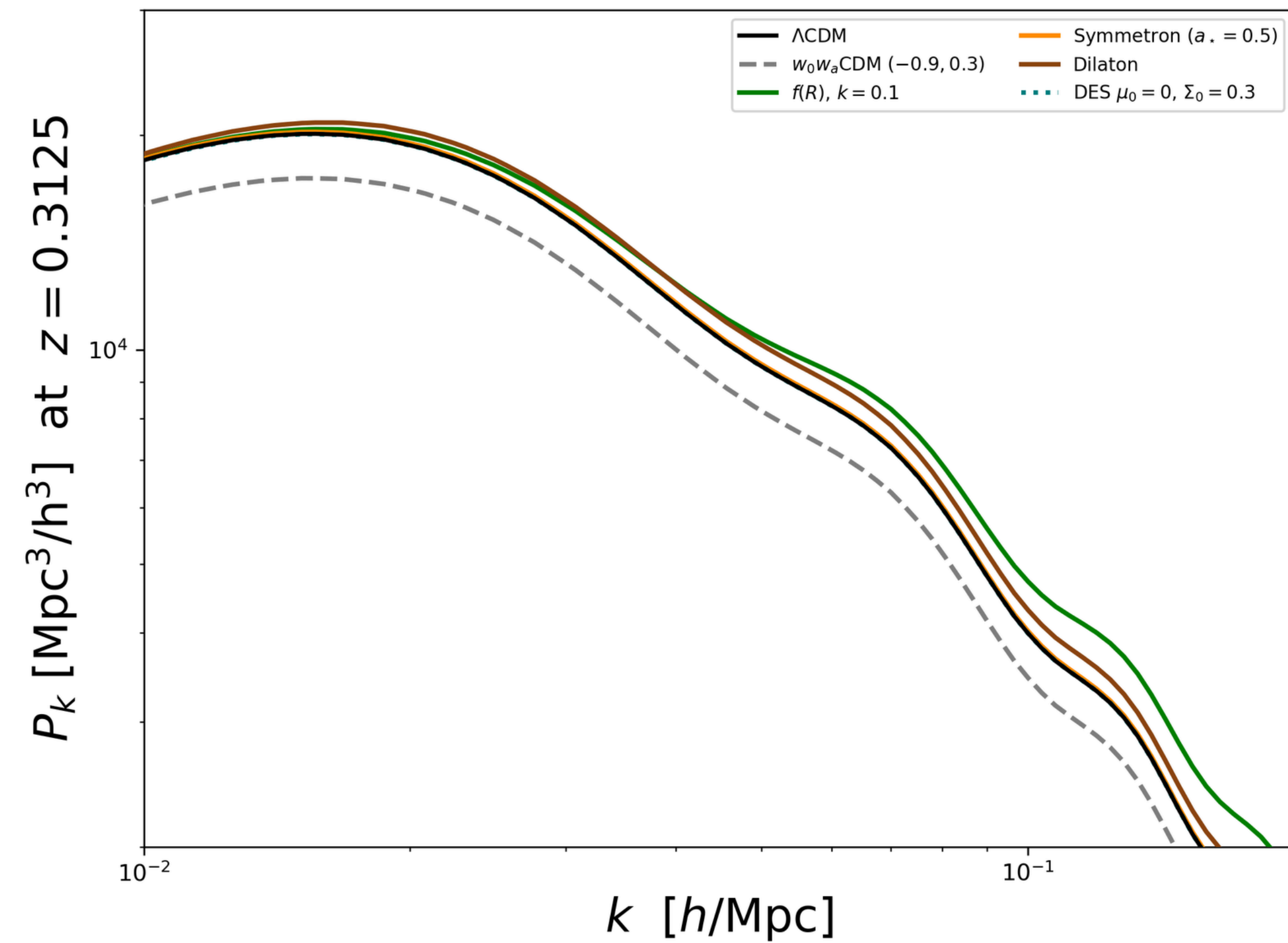
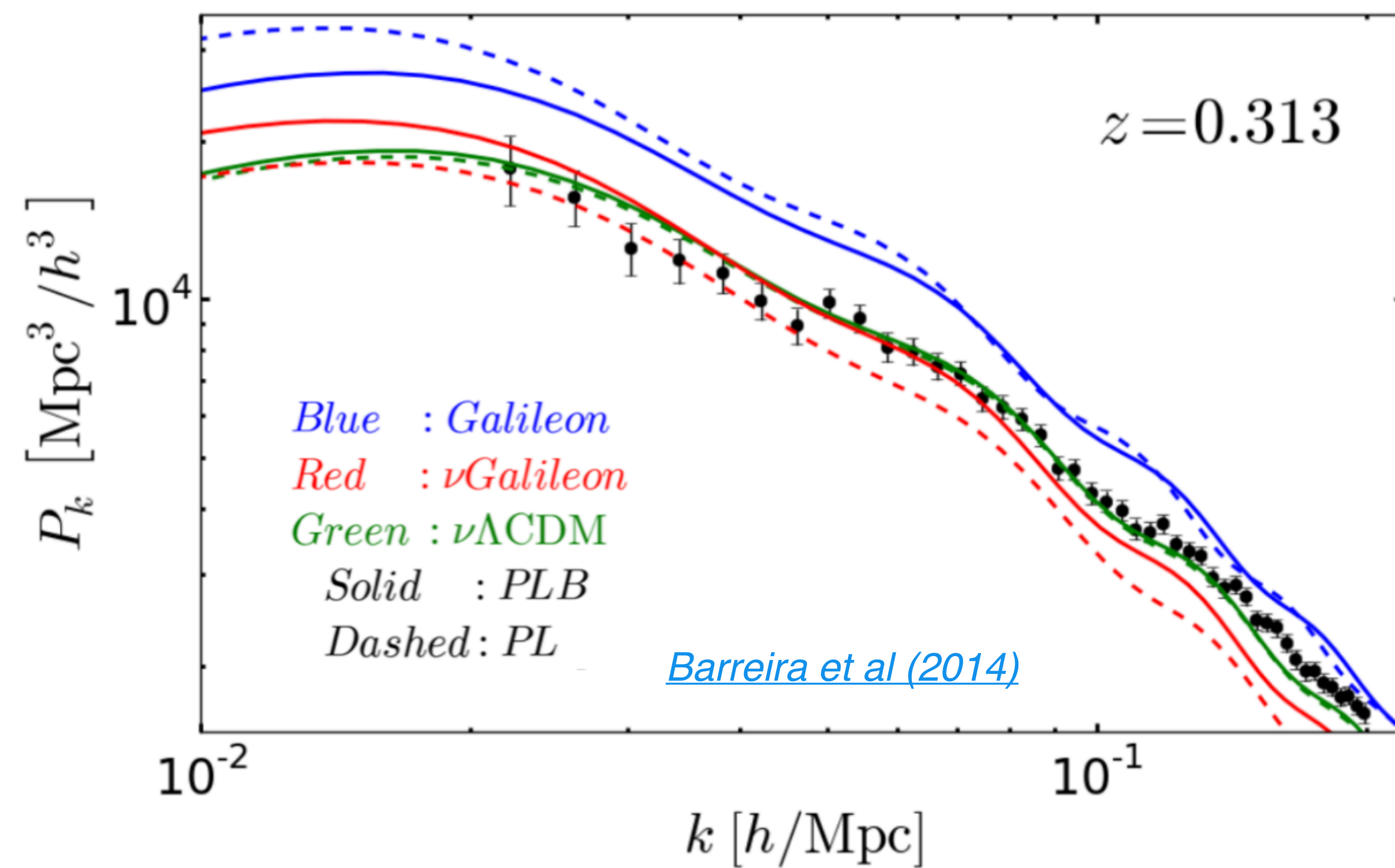
$$\ddot{\Delta}_m + 2H\dot{\Delta}_m - 4\pi G\mu(k, a)\bar{\rho}_m\Delta_m = 0$$

$$\Delta_m(a, k) = D(a)\Delta_{m,0}(k)$$



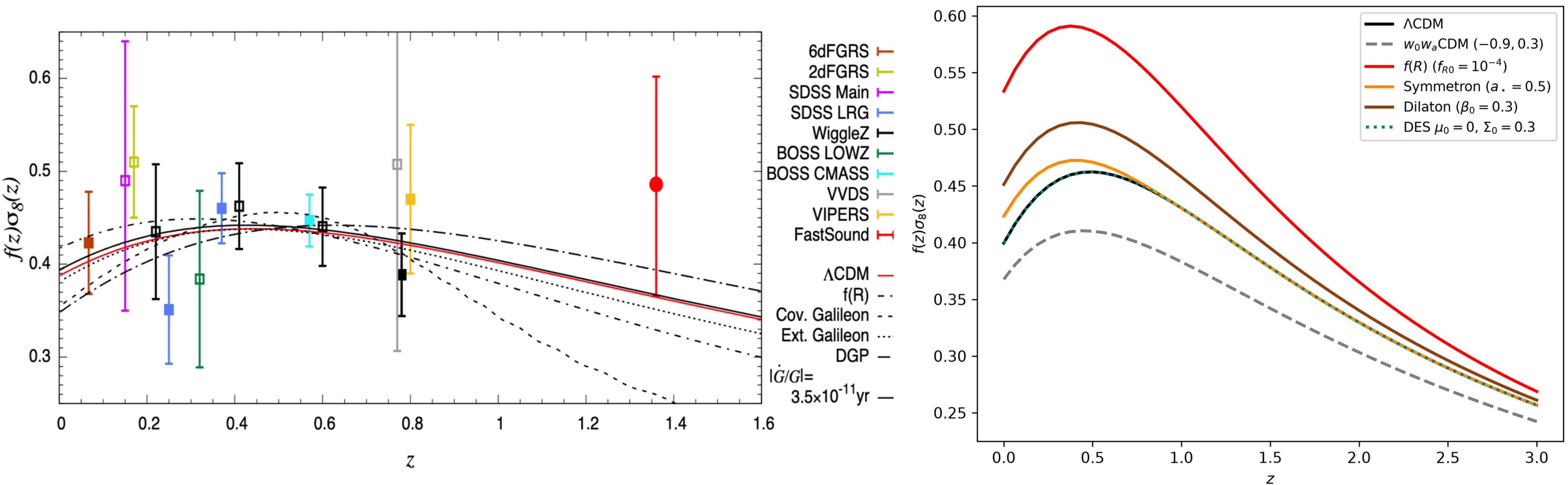


# $\mu$ and the growth of structures



# $\mu$ and the growth of structures

Surveys access  $f\sigma_8$  through measurements of the redshift space distortions (RSD), instead of  $f$  as it largely cancels the Galaxy bias





# Non linear growth

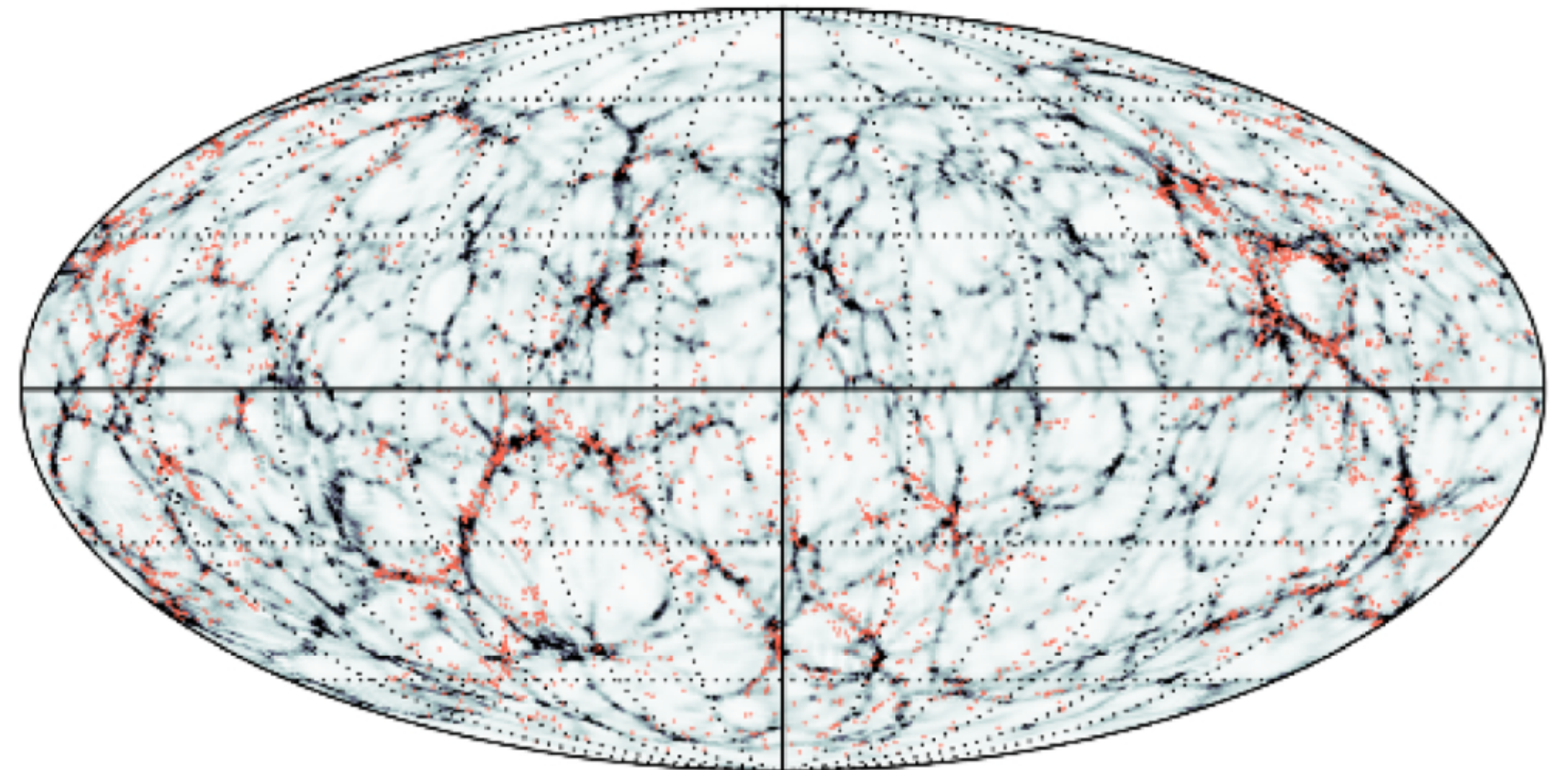
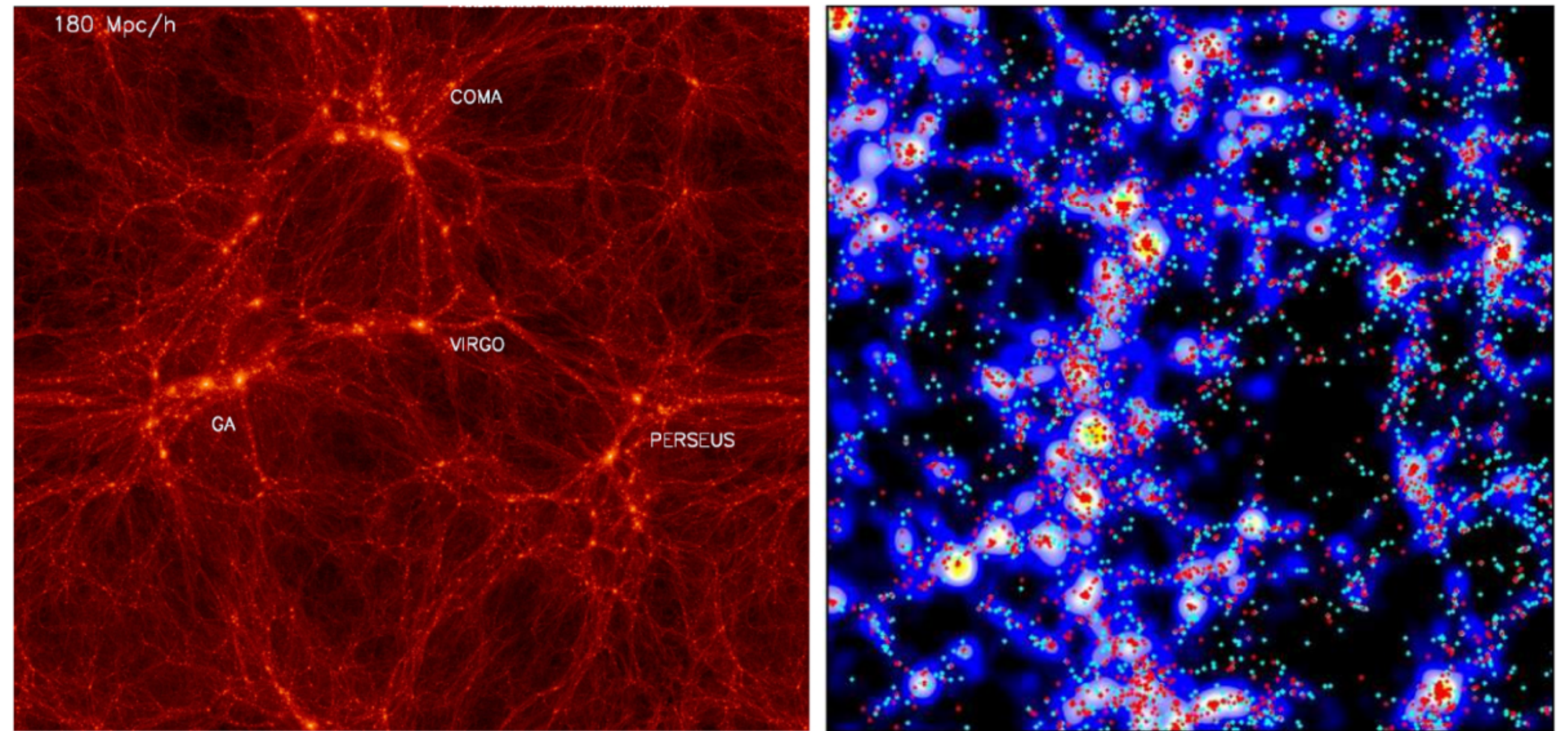
Above a certain size, perturbations are not linear anymore (first order is not enough)

You can try going to higher orders but at some point **non linear growth is terribly complex**.

You need **N-body simulations** which can be used to test gravity

For more, try:

<https://github.com/Hi-COLACode/Hi-COLA>

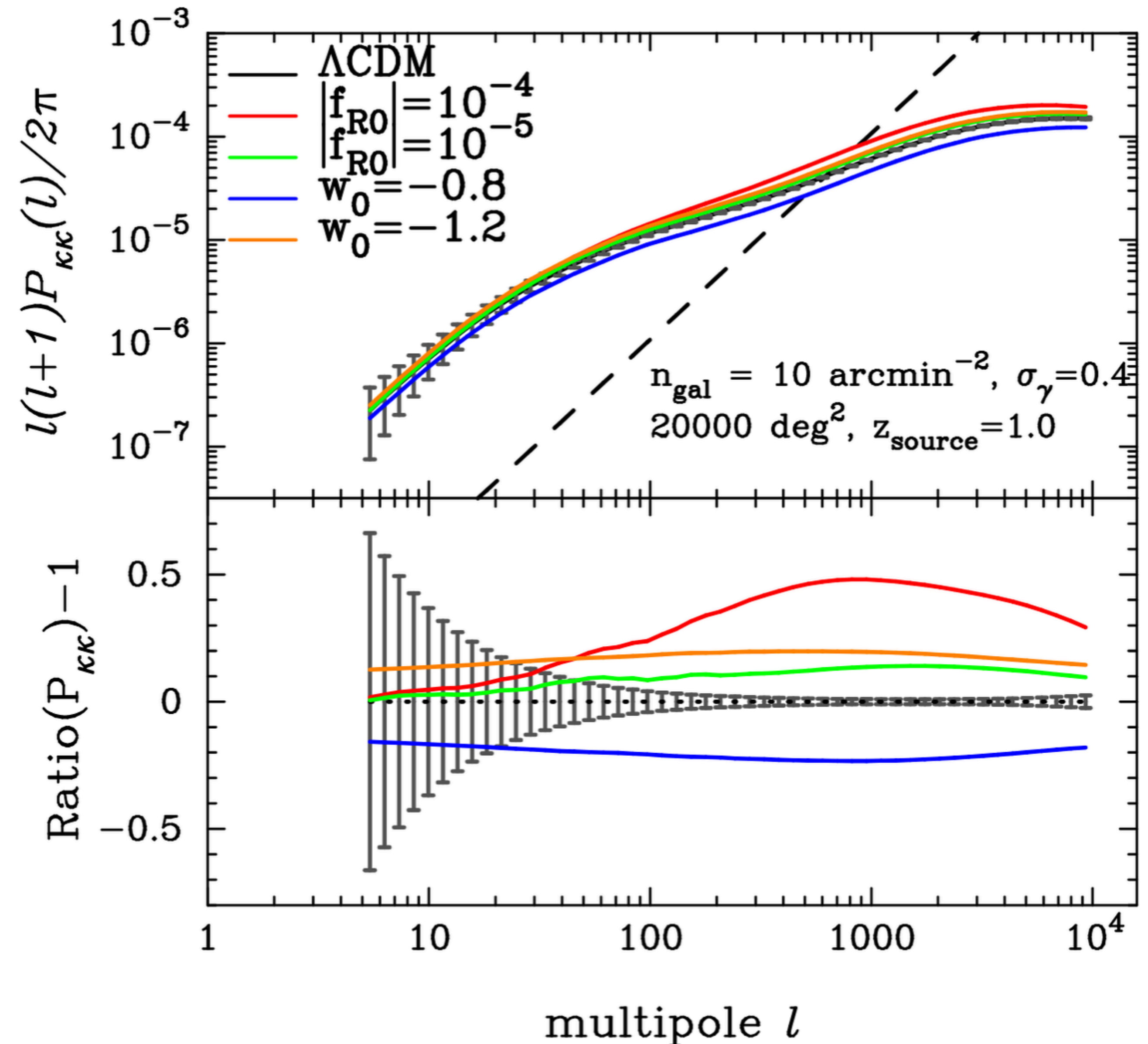




# $\Sigma$ and the lensing

Weak lensing (cosmic shear) measures  
among other things the convergence spectrum:

$$P_{\kappa\kappa}(\ell) = \int_0^{\chi_s} \frac{d\chi}{\chi^2} W(\chi)^2 \Sigma^2(a(\chi), k) P_m \left( k = \frac{\ell + \frac{1}{2}}{\chi}, z(\chi) \right)$$

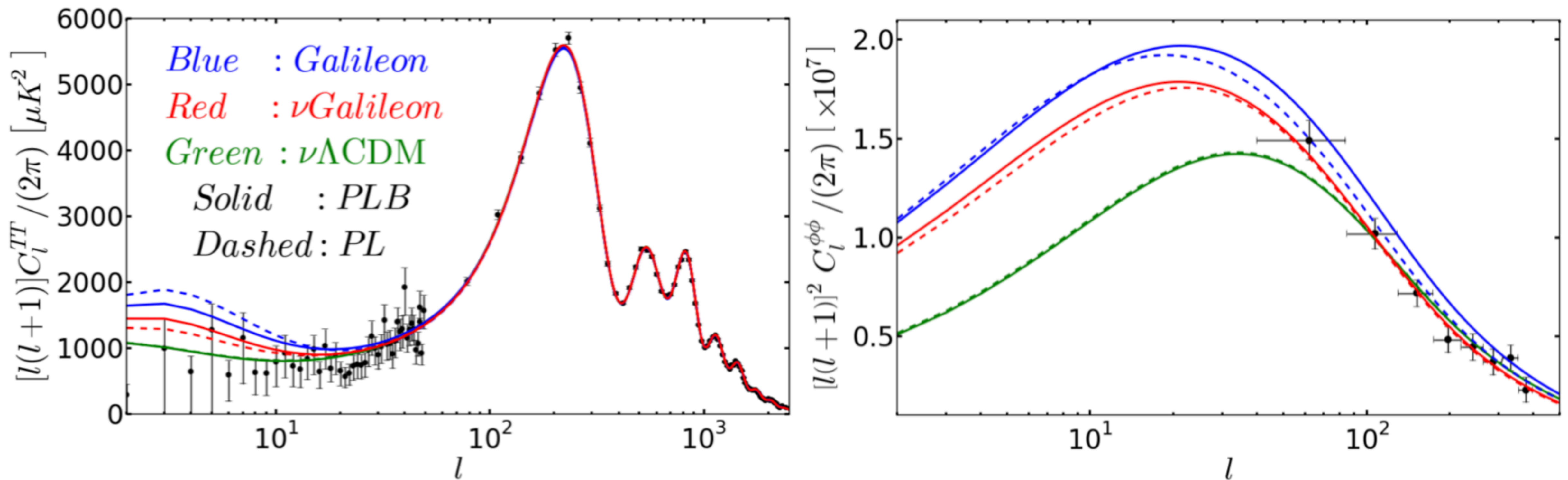




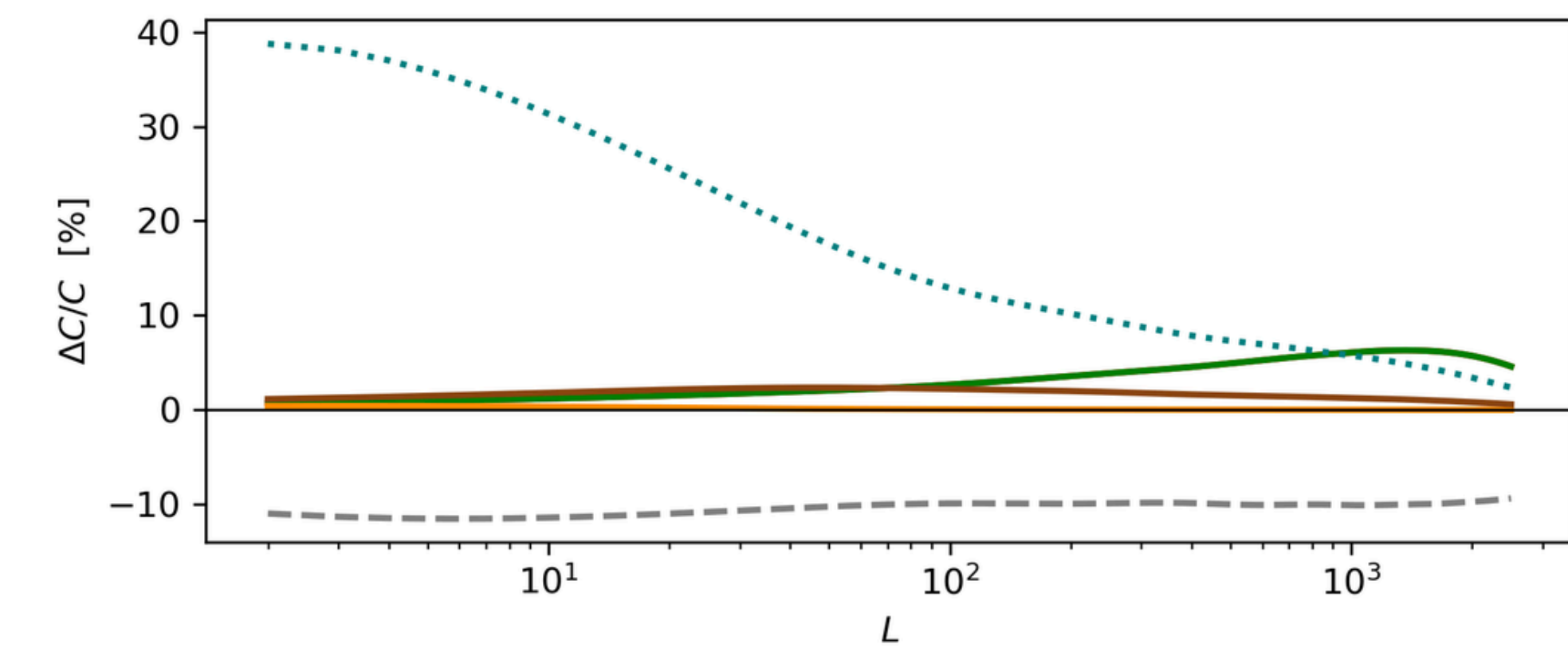
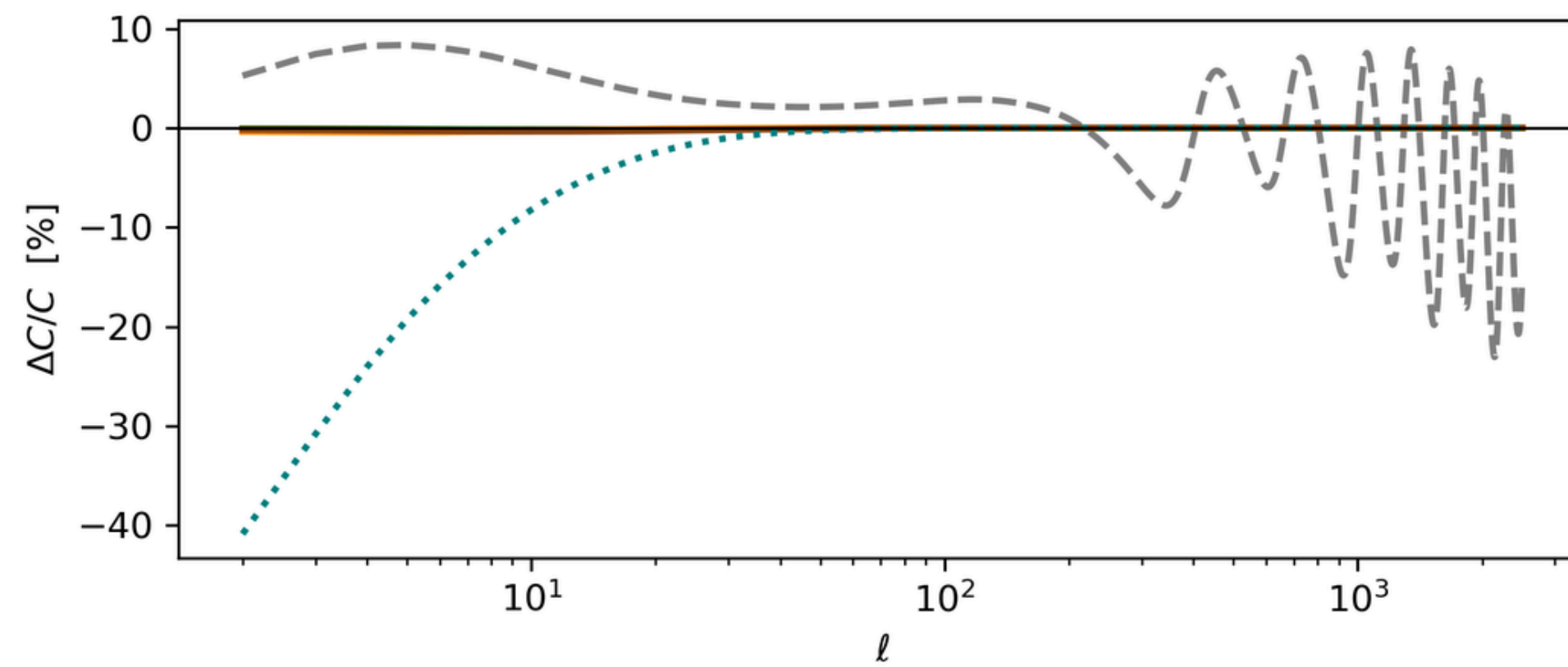
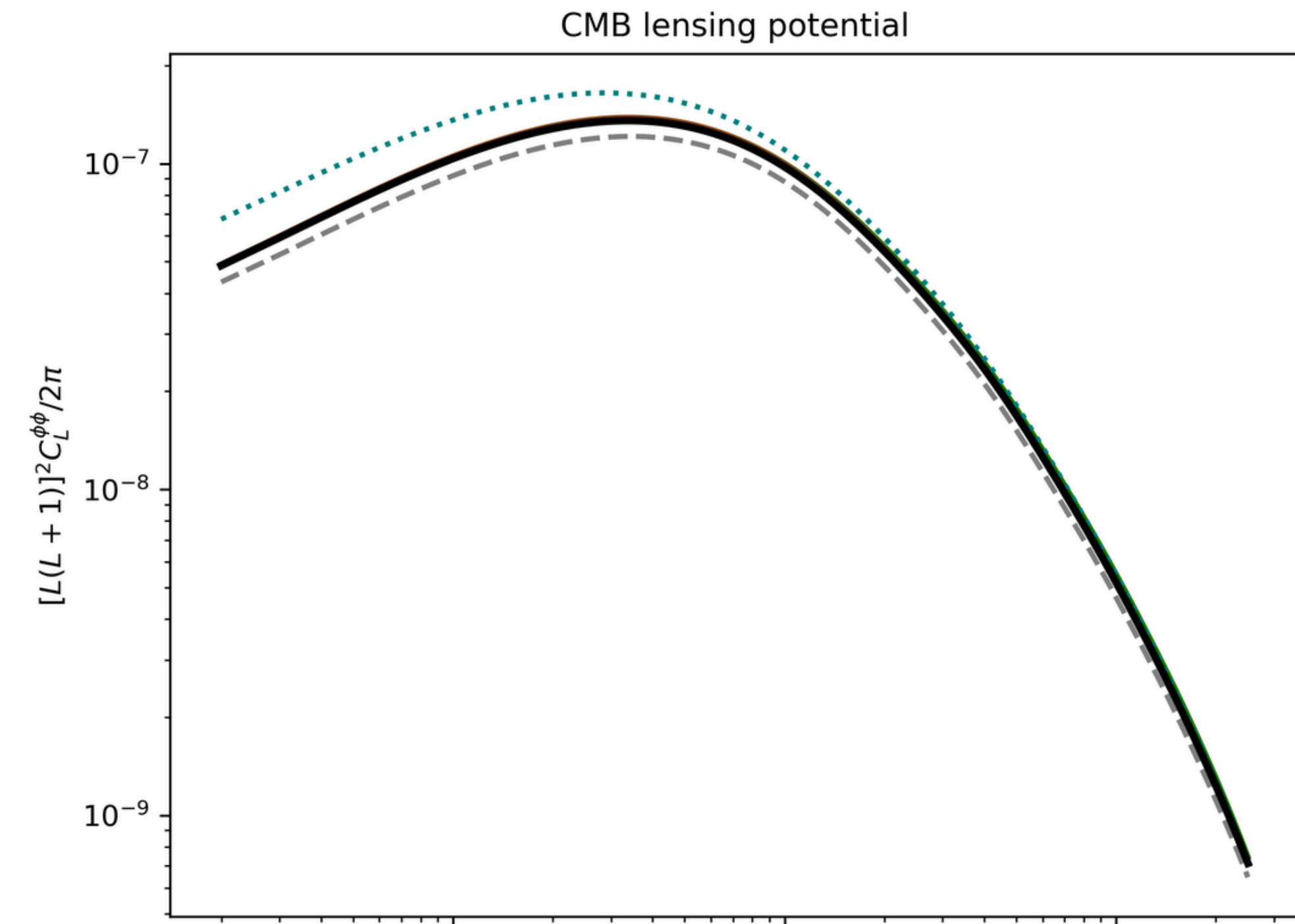
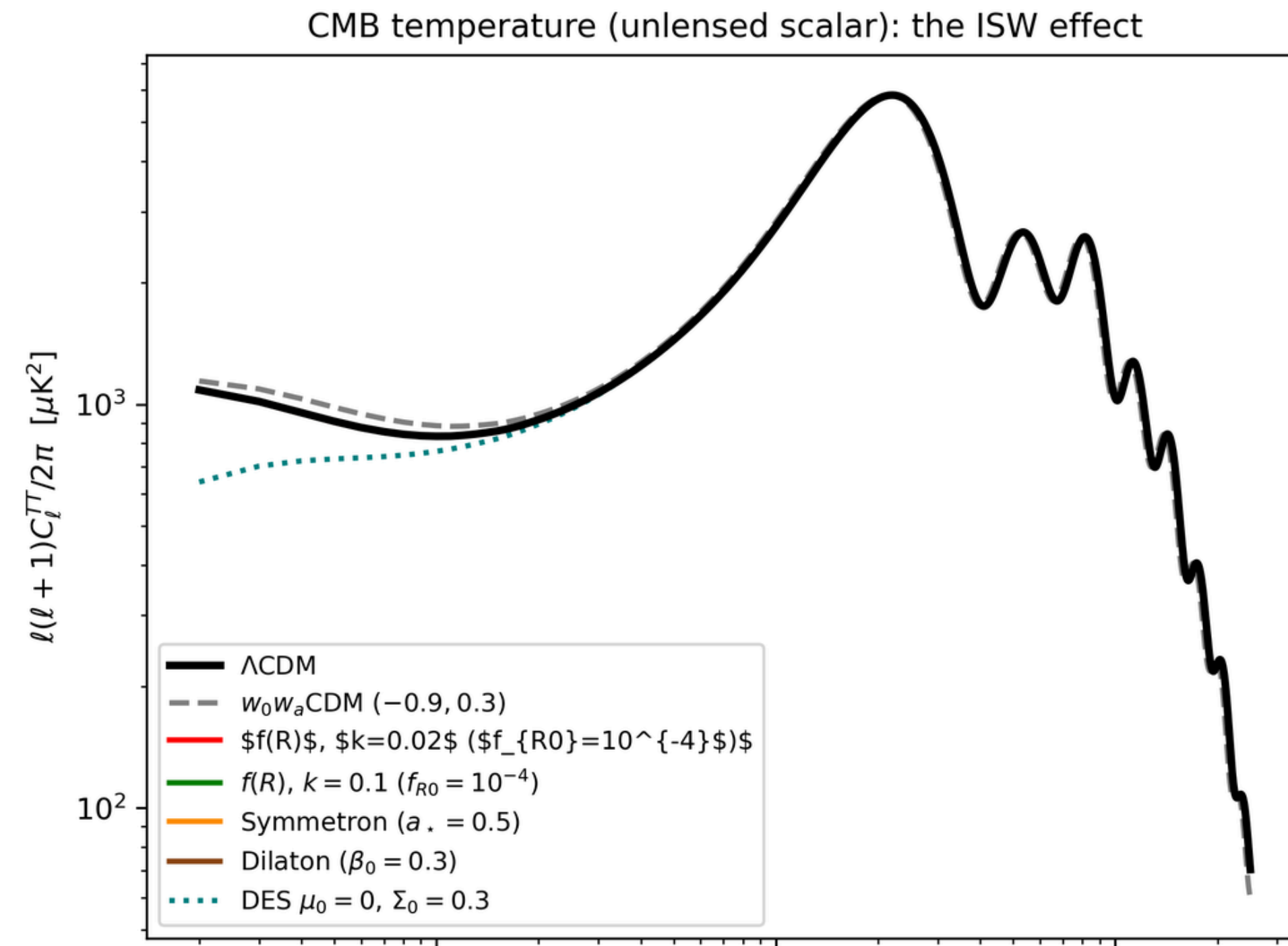
# $\Sigma$ and the CMB

Large scales of the CMB are sensitive to the ISW effect  $\propto \int (\dot{\Phi} + \dot{\Psi}) dt$

CMB is also sensitive to total light deflection by LSS (lensing)

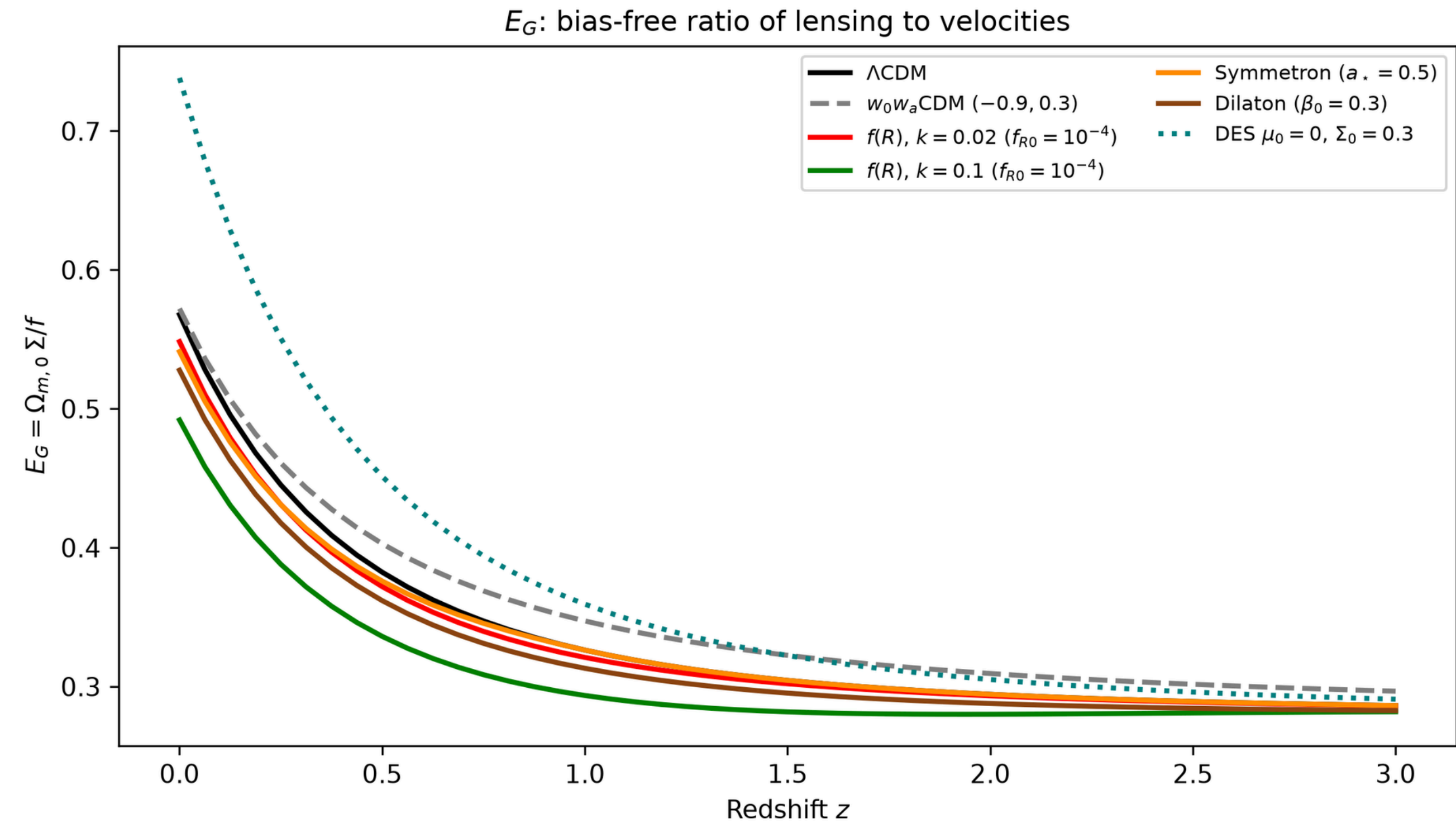
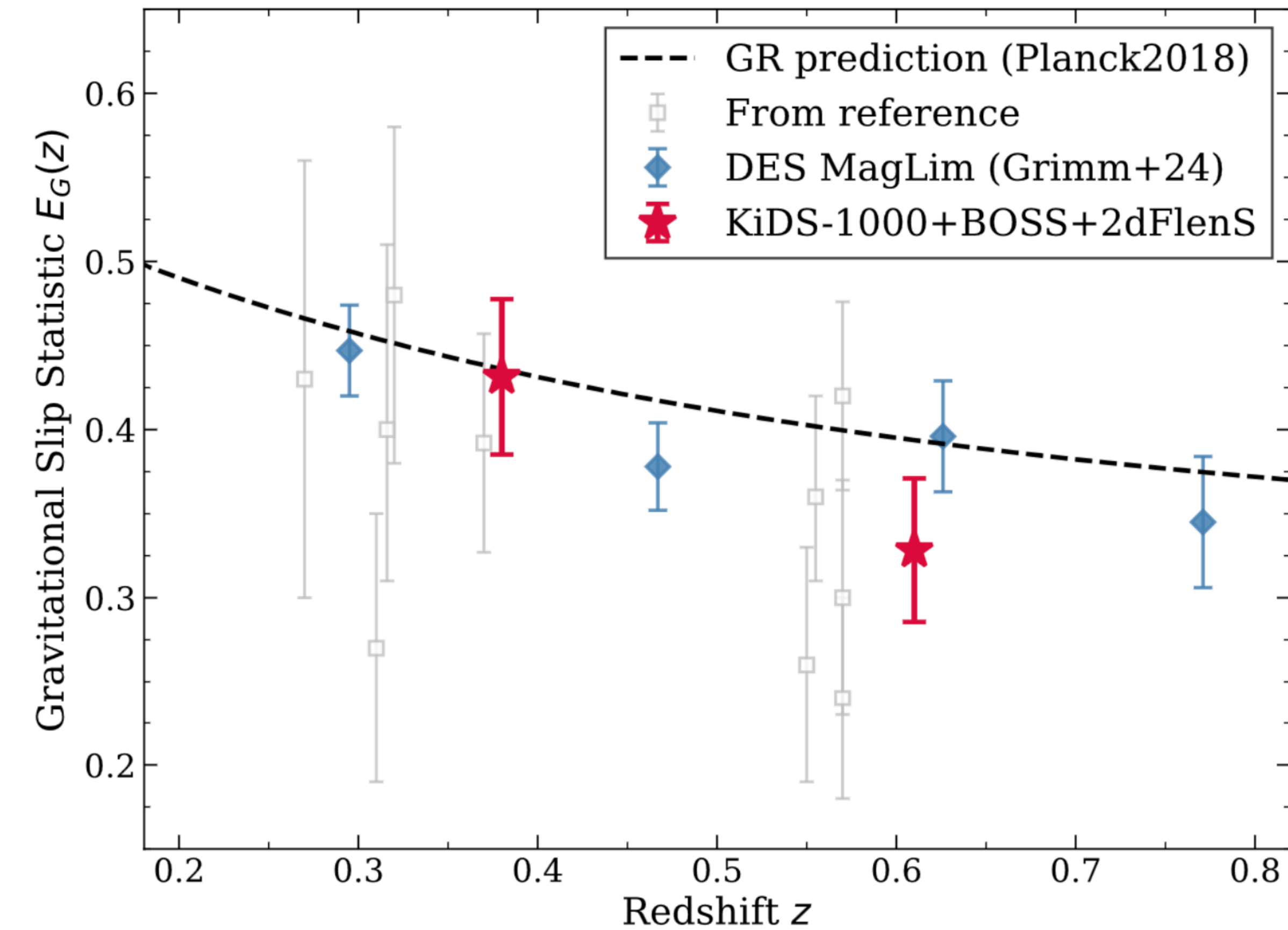


# $\Sigma$ and the CMB





# Combining all probes together to constraint modified gravity



$$E_G(k, z) = \frac{k^2(\Psi + \Phi)}{3H_0^2 a^{-1} f \delta_m}$$

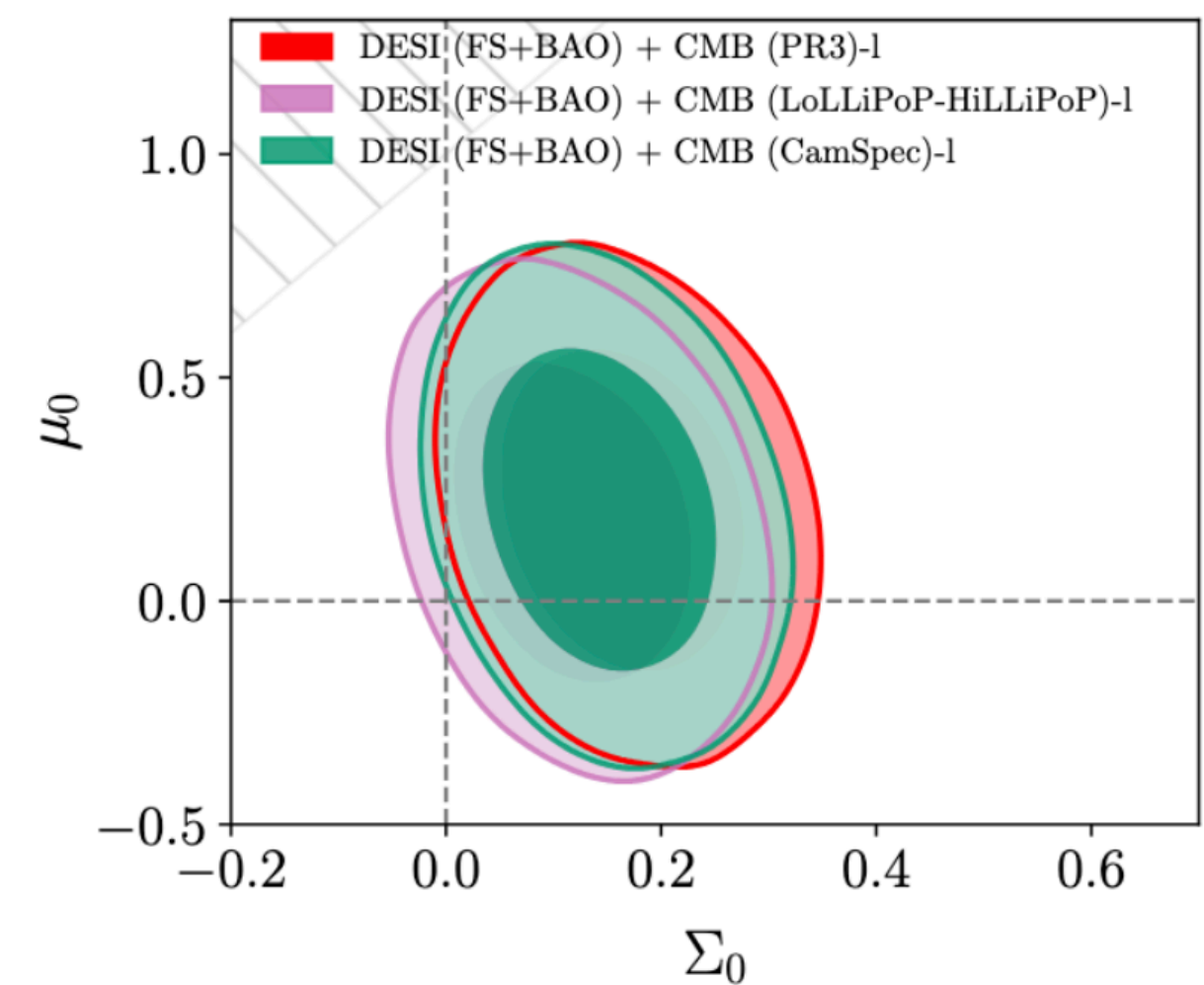
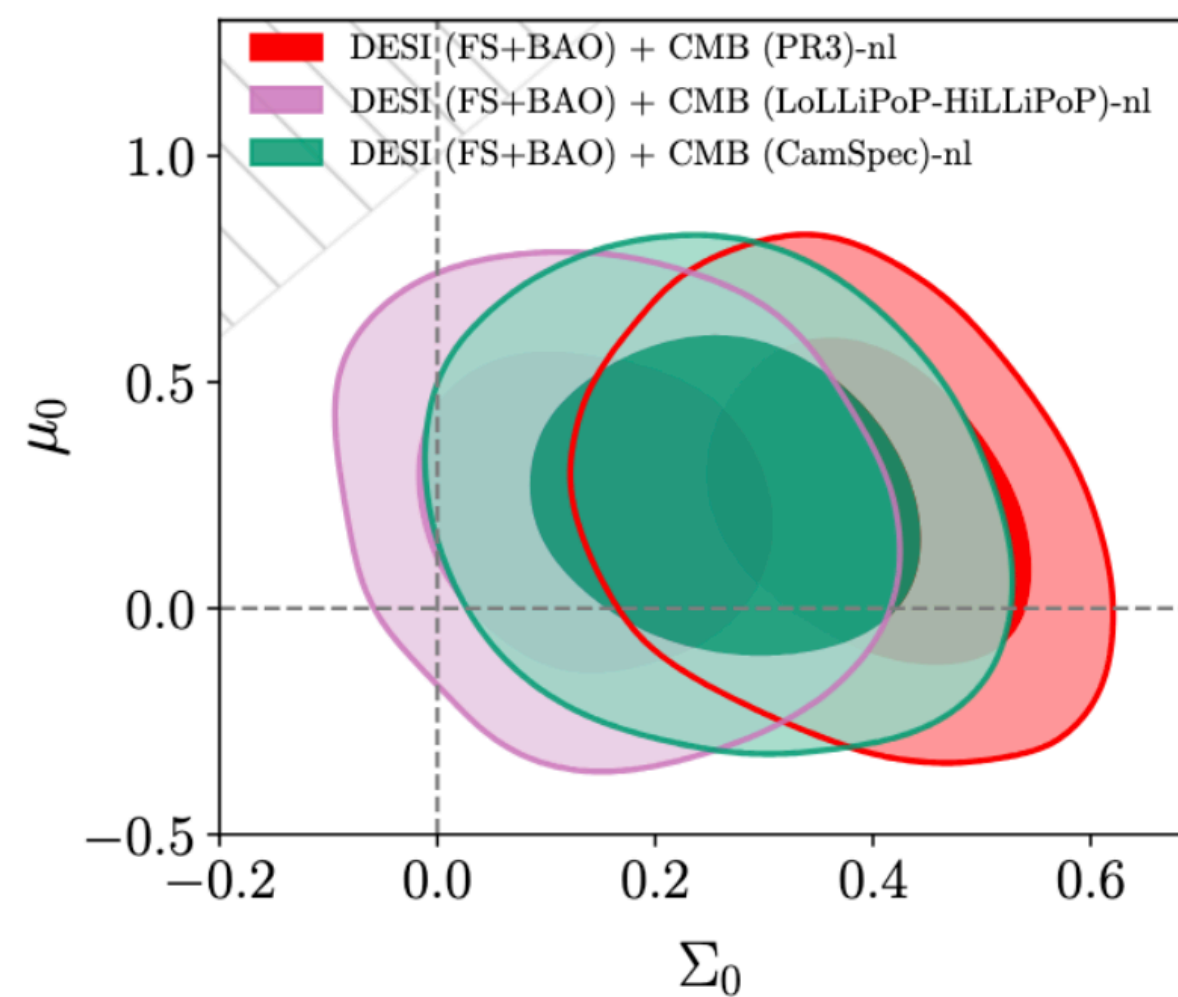
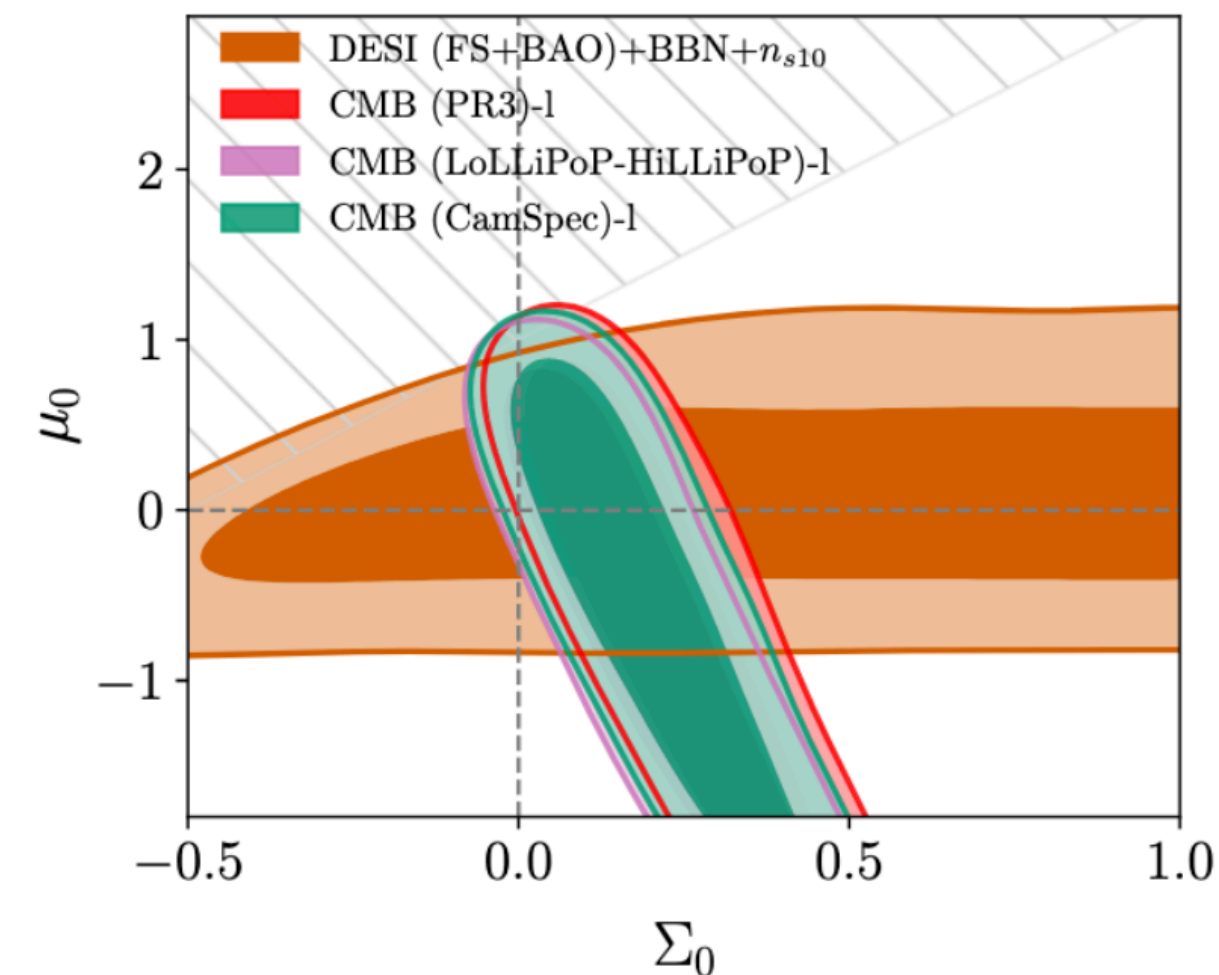
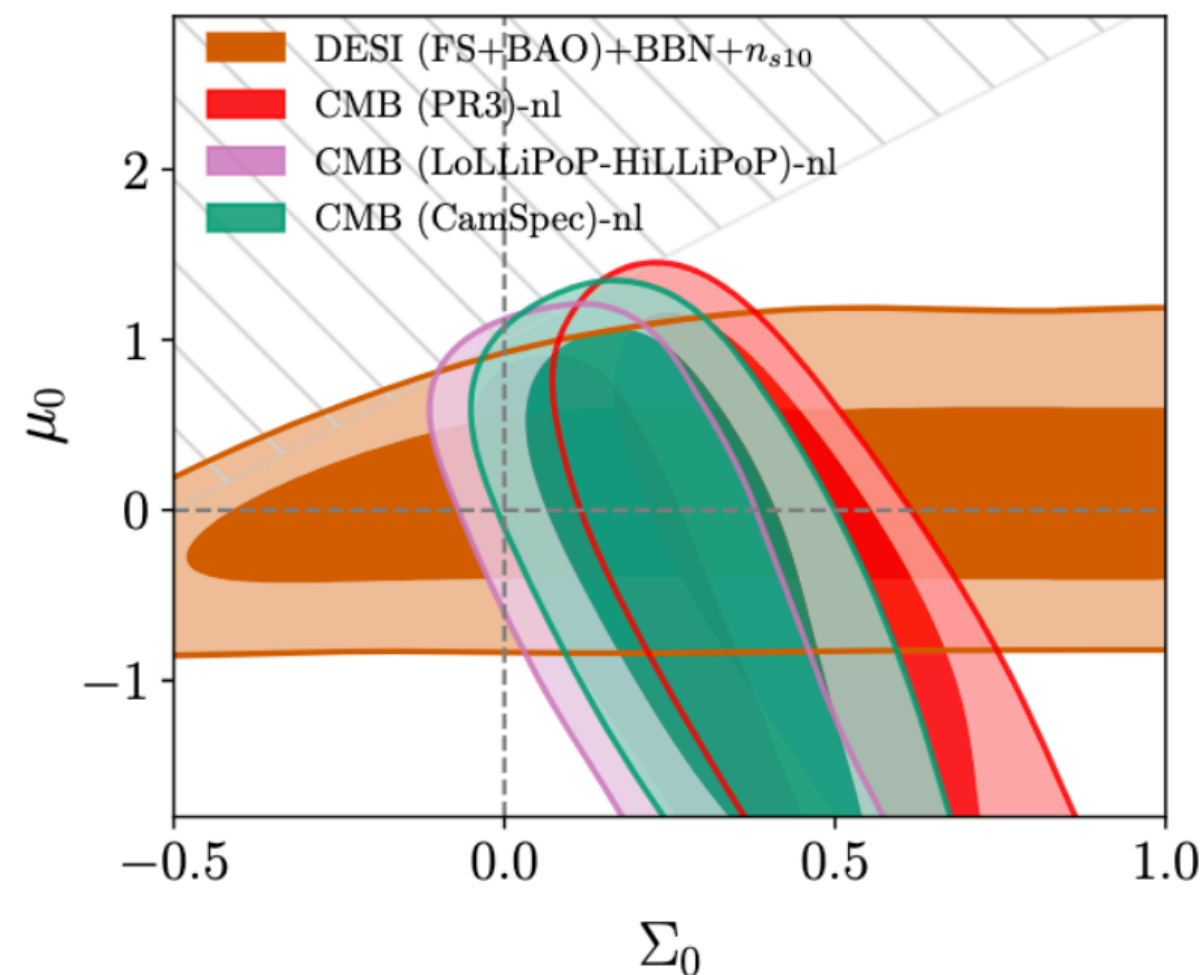
[Zhang \(2026\)](#)

# Combining all probes together to constraint modified gravity

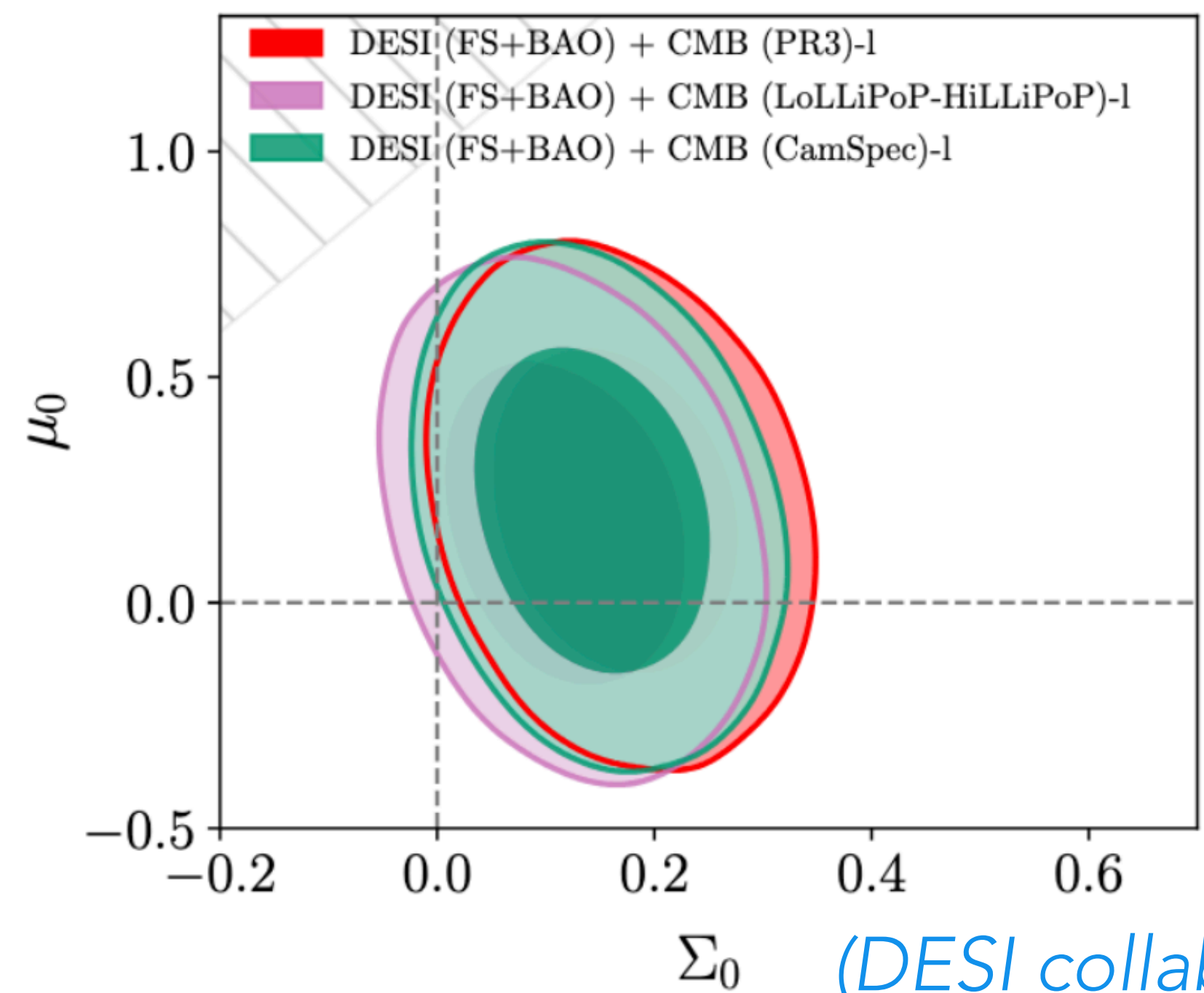
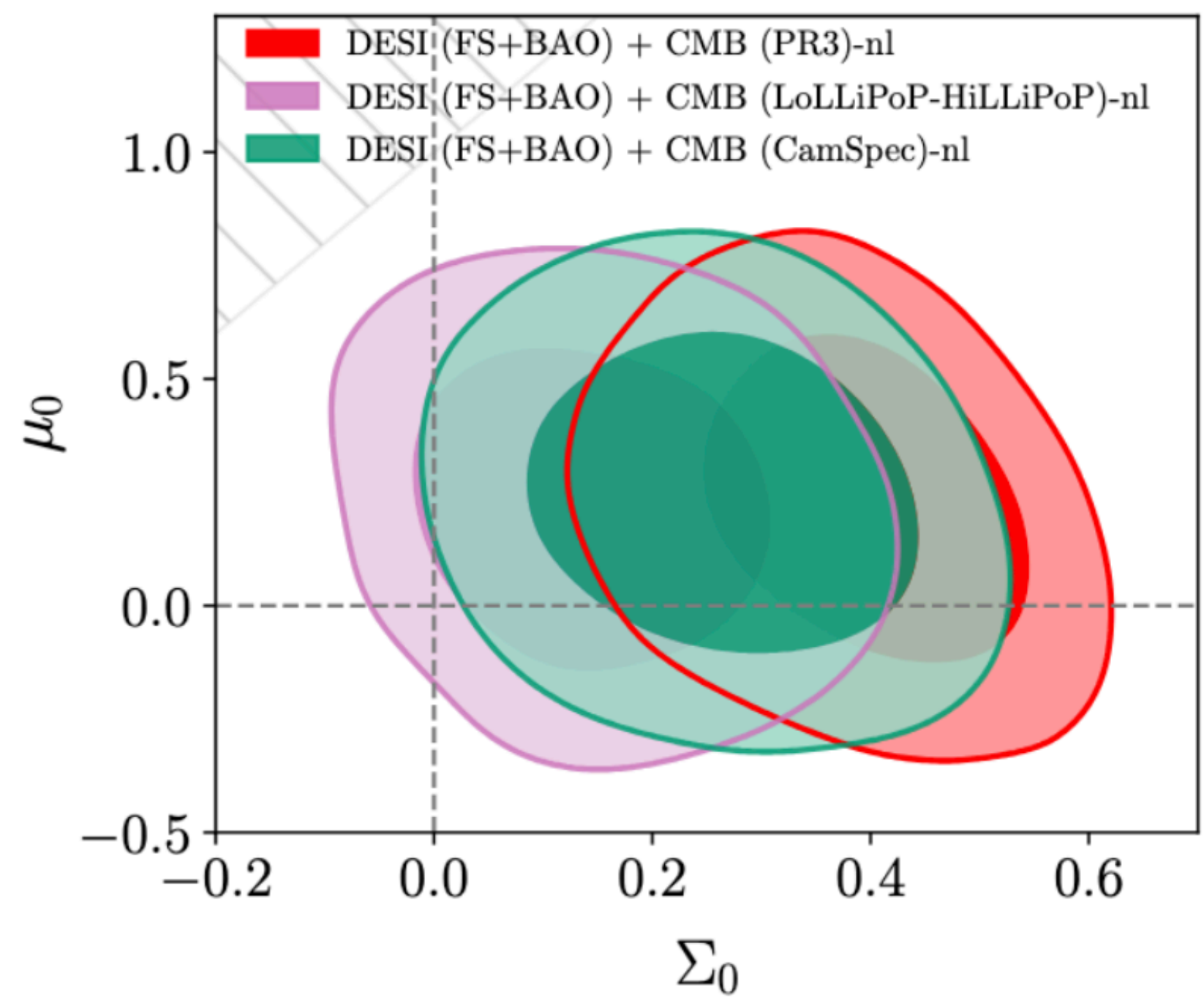
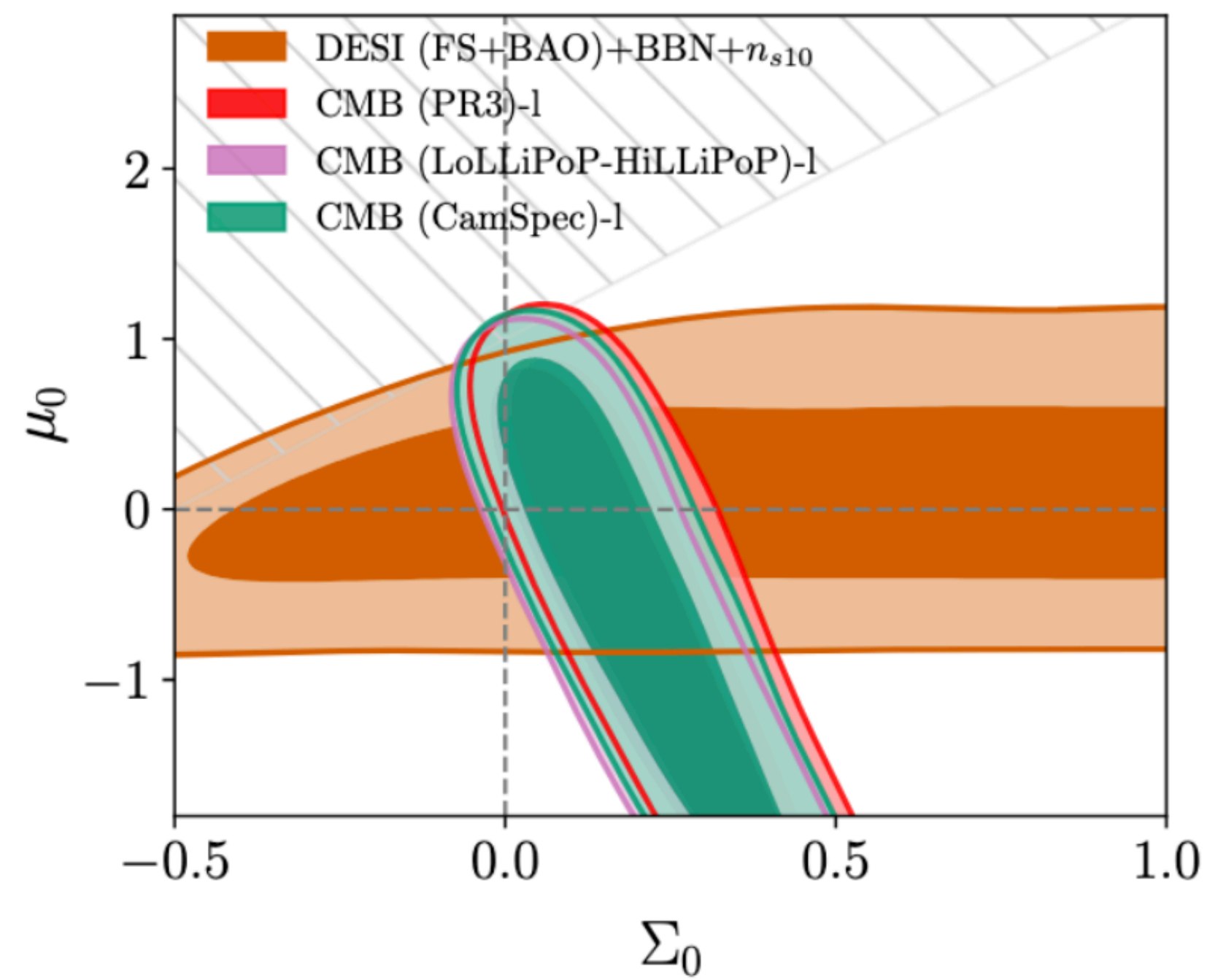
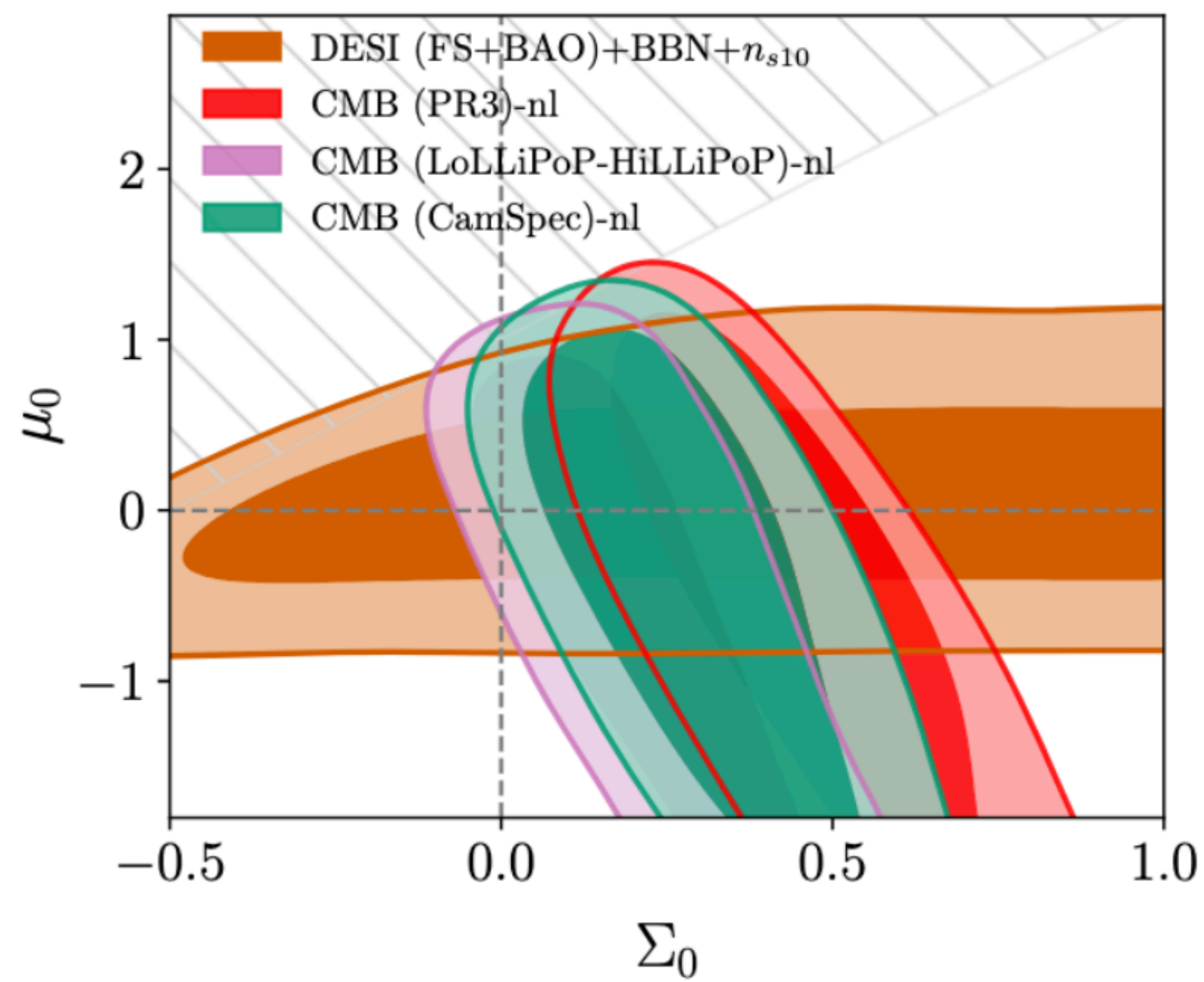
$$\mu(a, k) = 1 + \mu_0 \frac{\Omega_{\text{DE}}(a)}{\Omega_{\Lambda}} \left[ \frac{1 + c_1 (\lambda H(a)/k)^2}{1 + (\lambda H(a)/k)^2} \right]$$

$$\Sigma(a, k) = 1 + \Sigma_0 \frac{\Omega_{\text{DE}}(a)}{\Omega_{\Lambda}} \left[ \frac{1 + c_2 (\lambda H(a)/k)^2}{1 + (\lambda H(a)/k)^2} \right].$$

*(DESI collaboration (2024))*







(DESI collaboration (2024))

# Studying modified gravity is not an easy job!

1. What is **my theory**? Is it **motivated**?
2. Which **axiom of GR** (S1-S4, D1,D2) does it violate?
3. Does it have a **Lagrangian** if so, what is it?
4. Does it satisfies **EEP**, and **SEP**?
5. Compute the **equations of motion** for all the fields
6. Look at the **weak field** limit  $\rightarrow$  PPN parameters  $\rightarrow$  **constraints**
7. Are **constants varying**  $\rightarrow$  **constraints**
8. **Gravitational waves**  $\rightarrow c_T, \nu, m_g \rightarrow$  **constraints**
9. **Cosmological predictions**: background + perturbations  
 $\rightarrow H(z), \mu(k, z), \Sigma(k, z) \rightarrow$  **constraints**



A composite image featuring a snowy mountain peak in the foreground, with the Milky Way galaxy visible in the night sky above it. The galaxy's structure, including its spiral arms and central bulge, is clearly visible, with some areas appearing in red and blue. The mountain peak is illuminated by a warm, yellow light source, possibly the setting or rising sun, creating a silhouette effect. The overall scene is dark and atmospheric, with numerous stars visible in the background.

## II. Theories of modified gravity



# DISCLAIMER

- Lots of equations
- Probably lots of typo
- Do not try to understand all the equations if you can't
- Try not to confuse the forest for the trees, and get the general arguments
- All proof and details are in the notes and we can talk after the class



## First: The multiple faces of GR! (GR is fantastic)

- GR as a **field theory of a spin-2 graviton** in **flat** spacetime
- GR as a flat theory with **torsion** or **non-metricity**
- (Bonus: **Newtonian gravity** as curved space-time)

# GR: field theory or geometry?

A generic **field theory** in flat space-time is of the form:

$$\mathcal{L} = \mathcal{L}_K(\hat{\phi}) - V(\hat{\phi}) + \kappa \langle J, \hat{\phi} \rangle + \mathcal{L}_m$$

$$S = \int \mathcal{L} d^4x$$

Flat space-time  
action



$$\delta S = 0$$

Extremization



$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial(\phi)}$$

Euler Lagrange

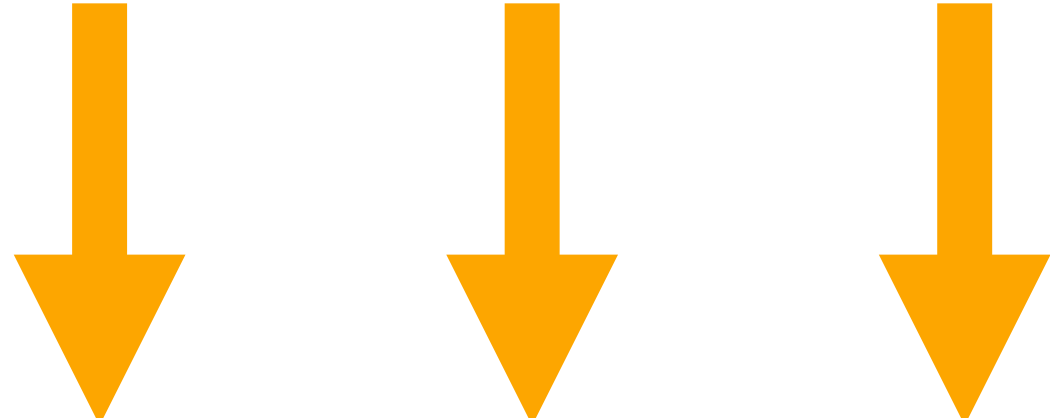


# GR: field theory or geometry?

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$$\mathcal{L} = \mathcal{L}_K(\hat{\phi}) - V(\hat{\phi}) + \kappa \langle J, \hat{\phi} \rangle + \mathcal{L}_m$$

The typical example being the **massive scalar field** theory:


$$\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + \kappa \phi J + \mathcal{L}_m(\psi)$$

With the **Klein-Gordon** equation of motion:

$$(\partial_\mu \partial^\mu \phi + m) = \kappa J$$

And the **mass** is defined generally as:

$$m = \frac{1}{2} \frac{\partial^2 V}{\partial \phi^2}$$

# GR: field theory or geometry?

A generic **field theory** in flat space-time is of the form:

$$\mathcal{L} = \mathcal{L}_K(\hat{\phi}) - V(\hat{\phi}) + \kappa \langle J, \hat{\phi} \rangle + \mathcal{L}_m$$

This does not look like GR at all! :(

$$\mathcal{L}_{EH} = \sqrt{-|g|} \left( \frac{1}{16\pi G} (R - 2\Lambda) + \mathcal{L}_m \right)$$

Of course not! GR is **about geometry** not about a field in flat space-time! **Or is it???**



# GR: field theory or geometry?

A generic **field theory** in flat space-time is of the form:

$$\mathcal{L} = \mathcal{L}_{\text{kin}}(\hat{\phi}) - V(\hat{\phi}) + \kappa \langle T, \hat{\phi} \rangle + \mathcal{L}$$

Stupid mindset I want you to have now:

This does not look

- Modern theories **are field theories on flat space-time** !  
GR is old and surely wrong. As modern physicists, let's try to write our own field theory of gravity which for sure would be much better!

Of course not! GR is **about geometry** not about a field in flat space-time! **Or is it???**

# Spin 0 gravitation: Nordstrom theory

The simplest Lagrangian for a scalar field transmitting gravity:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \kappa\phi\text{Tr}(T) + \mathcal{L}_m$$

([Nordström 1912-1913](#)), see also [Deruelle \(2011\)](#)



Simplest interaction you can think of that would make sense:

- $\phi$  should interact universally with something all matter have. We know energy density (mass) and pressure gravitates  $\rightarrow$  it has to be  $T_{\mu\nu}$  somehow
- It should be linear in  $T$  and  $\phi$  otherwise **non additive charges** and **matter without field**
- it should be a Lorentz scalar, linear from  $T_{\mu\nu}$  we have only  $\mathcal{T} = \text{Tr}(T) = T^\mu_\mu$



# Spin 0 gravitation: Nordstrom theory

The simplest Lagrangian for a scalar field transmitting gravity:

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Static field Green function/integration

$$\partial_\mu\partial^\mu\phi = -\kappa\text{Tr}(T) \quad \longrightarrow \quad \Delta\phi = \kappa\rho \quad \xrightarrow{\rho = m_1\delta(x_0 - x)} \quad \phi = -\frac{\kappa}{4\pi}\frac{m_1}{r}$$

$V = m_2\phi$   
Coupling to another mass

$$V_{\text{int}} = -\frac{\kappa^2}{4\pi}\frac{m_1m_2}{r}$$

$$G = \frac{\kappa^2}{4\pi}$$

# Spin 0 gravitation: Nordstrom theory

The simplest Lagrangian for a scalar field transmitting gravity:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \kappa\phi\text{Tr}(T) + \mathcal{L}_m$$

Great! This is a relativistic version of Newton theory, it fits all our requirements of lecture 1 !

**EVEN BETTER**: It contains **weak equivalence principle, EEP and SEP**! (Might be the only one with GR)\*

**So what's the problem?**

\* Under the natural replacement  $\kappa\phi\text{Tr}(T) \equiv \mathcal{L}_m[(1 + \kappa\phi)^2\eta_{\mu\nu}]$ , see conditions d'achat en magasin



# Spin 0 gravitation: Nordstrom theory

The simplest Lagrangian for a scalar field transmitting gravity:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \kappa\phi\text{Tr}(T) + \mathcal{L}_m$$

Great! This is a relativistic version of Newton theory, it fits all our requirements of lecture 1 !

**It bends no light!**

$$\text{Tr}(T) = -\rho + 3p = 0 \quad \text{For radiation} \quad \text{Bad for cosmology!}$$

**It predicts wrong perihelion shift!**

**(+ Gravitational waves?)**

# Spin 0 gravitation: Nordstrom theory = geometry !

Define a metric (that is a **conformal rescaling** of a flat metric):

$$g_{\mu\nu} = (1 + \kappa\phi)^2 \eta_{\mu\nu}$$

Nordstrom theory is absolutely equivalent to the **geometric relation**: ([\*Einstein and Fokker 1914\*](#))

$$R = 24\pi G \text{Tr}(T)_g$$

(Light is not affected as conformal transformations leave Maxwell Lagrangian invariant)

\* Under the natural replacement  $\kappa\phi \text{Tr}(T) \equiv \mathcal{L}_m[(1 + \kappa\phi)^2 \eta_{\mu\nu}]$ , see conditions d'achat en magasin



# Other spins

- **Spin 1/2: Can't work!** Single exchange would be scattering, pair exchange give a force in  $1/r^5$
- **Spin 1: Can't work!** Opposite repels instead of attracting! Cf electromagnetism
- **Spin  $\geq 3$ : Can't work!** No conserved current (general arguments based on Noether second theorem and possible conserved sources) ([Weinberg+](#))

See the notes + [Feynman's](#) lectures on gravitation and [Weinberg](#) quantum theory of fields

# Spin 2 gravitation: Fierz-Pauli action

The **unique Lorentz-invariant, two-derivative, ghost-free** kinetic term is

$$\mathcal{L}_{\text{FP}} = -\frac{1}{2}\partial_\kappa h_{\mu\nu}\partial^\kappa h^{\mu\nu} + \partial_\mu h_{\nu\kappa}\partial^\nu h^{\mu\kappa} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\kappa h\partial^\kappa h$$

[Fierz and Pauli \(1939\)](#)



# Spin 2 gravitation: Fierz-Pauli action

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$$+\frac{\kappa}{2}h_{\mu\nu}T^{\mu\nu}$$

Natural **interaction** to source **gravity** (1/2 is a convention)

(**WEP** naturally built in, from field considerations on allowed couplings, see Weinberg again)

# Spin 2 gravitation: Fierz-Pauli action

The **unique Lorentz-invariant, two-derivative, ghost-free** kinetic term is

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Equations of motions give exactly (linearised) **Einstein equations** (D1) with  $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ !

Predicts:

$$V_{\text{int}} = \frac{-Gm_1m_2}{r} \quad \delta\theta = \frac{4GM}{c^2b} \quad \text{Fantastic!}$$



# Spin 2 gravitation: Fierz-Pauli action

The **unique Lorentz-invariant, two-derivative, ghost-free** kinetic term is

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Equations of motions give exactly (linearised) **Einstein equations** (D1) with  $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ !

Do not predict: correct mercury perihelion shift (non-linear GR)

# Spin 2 gravitation: Fierz-Pauli action

The **unique Lorentz-invariant, two-derivative, ghost-free** kinetic term is

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What is missing: **strong equivalence principle**!

**Gravity bootstrap**  Gives **uniquely GR**! [Deser \(1970\)](#)

So GR can be understood as a **field theory** (bootstrapped) of spin-2 field: the **graviton**

Field and geometry are two faces of the same coin!



# Spin 2 gravitation: Fierz-Pauli action

The **unique Lorentz-invariant, two-derivative, ghost-free** kinetic term is

$$\mathcal{L}_{\text{FP}} = -\frac{1}{2}\partial_\kappa h_{\mu\nu}\partial^\kappa h^{\mu\nu} + \partial_\mu h_{\nu\kappa}\partial^\nu h^{\mu\kappa} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\kappa h\partial^\kappa h \quad \text{Fierz and Pauli (1939)}$$

$$+\frac{\kappa}{2}h_{\mu\nu}(T^{\mu\nu} + T_{(h)}^{\mu\nu} + \dots) \\ +M_g^2 \left( h_{\mu\nu}h^{\mu\nu} - (\eta^{\mu\nu}h_{\mu\nu})^2 \right)$$

## Massive gravity (level 0)

Could explain cosmic acceleration?  
(Only term without ghost, dof  $2 \rightarrow 5$   
diff. gauge symmetry broken)

Recall from first class:  $M_g \leq 1.27 \times 10^{-23} \text{ eV}/c^2$

For cosmic exp:  $M_g \simeq 1 \times 10^{-33} \text{ eV}/c^2$

Massive gravity is a super exciting topic, we will not have time to properly cover ... see notes or the wonderful Video classes "Modified gravity by [Rachel Rosen](#)"

# Spin 2 gravitation: Fierz-Pauli action

The **unique Lorentz-invariant, two-derivative, ghost-free** kinetic term is

$$\mathcal{L}_{\text{FP}} = -\frac{1}{2}\partial_\kappa h_{\mu\nu}\partial^\kappa h^{\mu\nu} + \partial_\mu h_{\nu\kappa}\partial^\nu h^{\mu\kappa} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\kappa h\partial^\kappa h$$

$$+\frac{\kappa}{2}h_{\mu\nu}(T^{\mu\nu} + T_{(h)}^{\mu\nu} + \dots)$$

$$+M_g^2\left(h_{\mu\nu}h^{\mu\nu} - (\eta^{\mu\nu}h_{\mu\nu})^2\right)$$

## Massive gravity (level 1)

**vDVZ discontinuity:** Does not give GR when  $M_g \rightarrow 0$

**Vainshtein screening:** Hides the discontinuity

**dRGT gravity:** Modern form of the theory

Massive gravity is a super exciting topic, we will not have time to properly cover ... see notes or the wonderful Video classes "Modified gravity by [Rachel Rosen](#)"



# A sidetone I can't resist: Newton-Cartan theory

A Newtonian space-time is the quadruplet

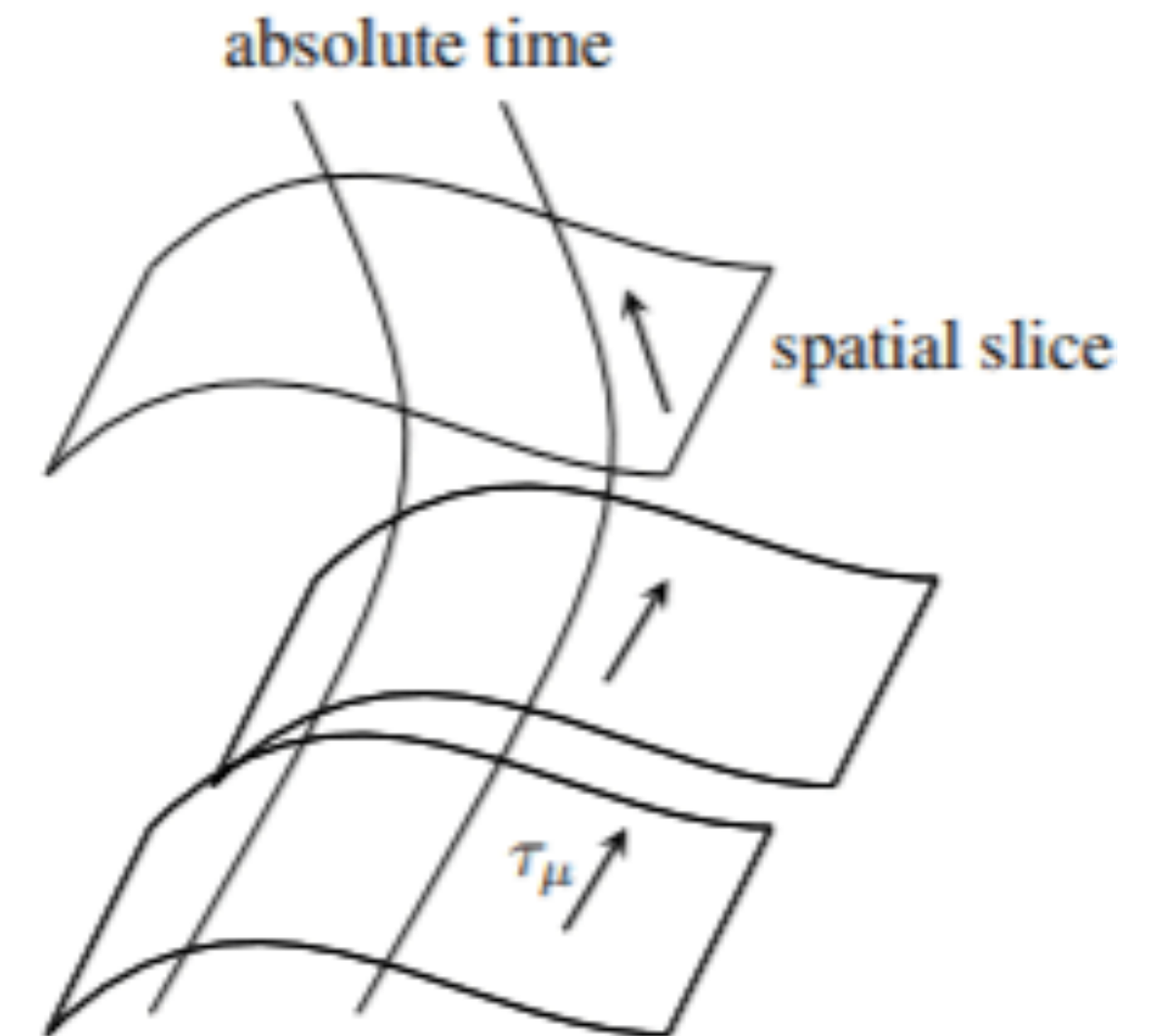
$$(M, \tau, k, \nabla)$$

General torsion-free connection

Absolute-time  $\tau_\mu = \partial_\mu t$   
(Clock 1-form)

Space metric  
 $h^{\mu\nu}$

5 types of axioms



[Malament \(2012\)](#)

[Schwartz \(2023\)](#)

[Schuller YT videos \(2015\)](#)

# A sidetone I can't resist: Newton-Cartan theory

A Newtonian space-time is the quadruplet  $(M, \tau, k, \nabla)$

## D types of axioms:

Newton theory can be rewritten **fully geometrically**:

- **Newton first law**: all free bodies move on geodesic of the connection  
(gravity is there):

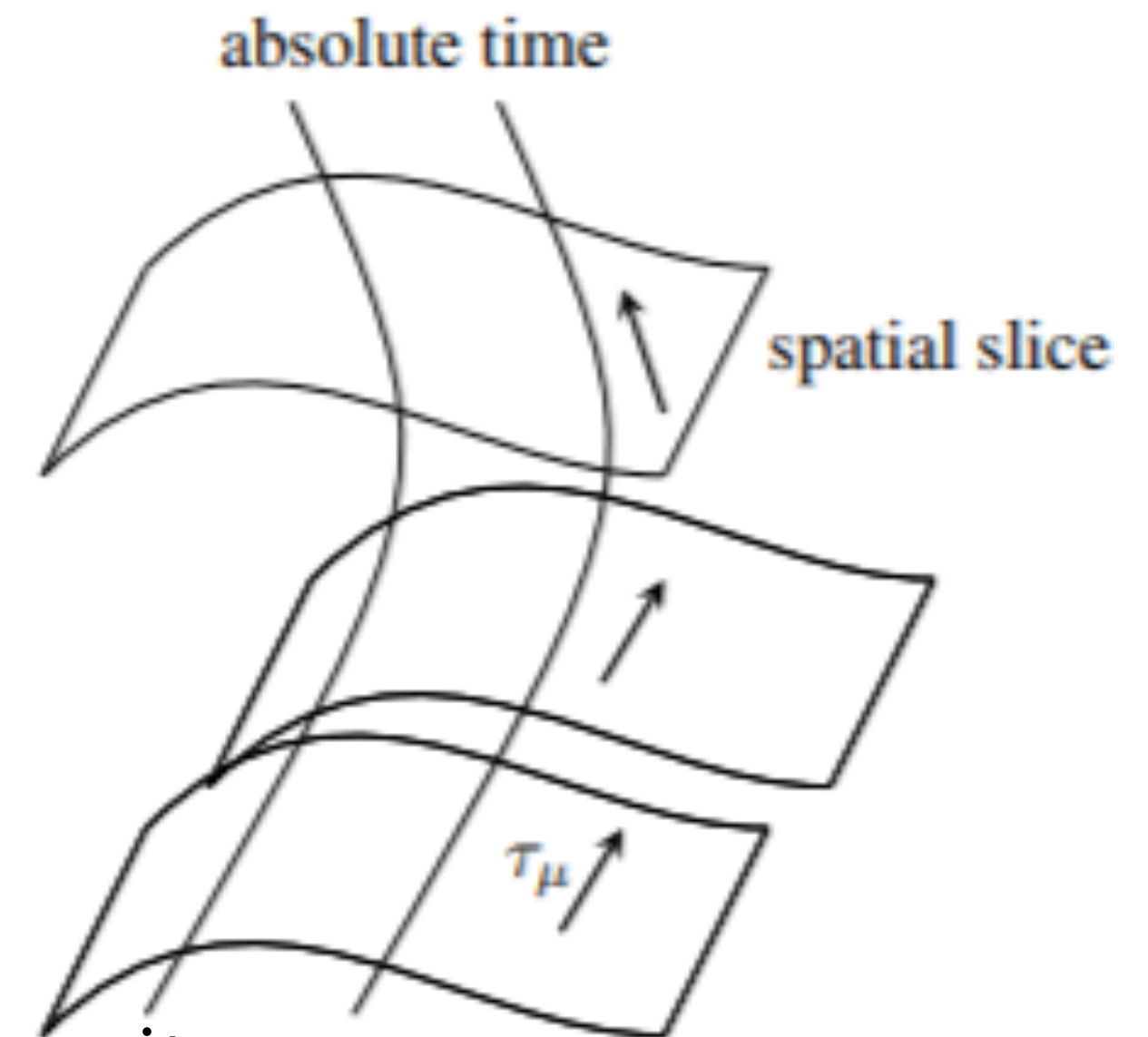
$$\nabla_v v = 0 \quad \tau(v) > 0$$

- **Newton second law**: if a body deviates from a geodesic a Force  $F$  is exerted on it:

$$\nabla_v v = F/m \quad \tau(F) = 0$$

- Or in the single **Poisson equation**:

$$R_{\mu\nu} = 4\pi G\rho\tau_\mu\tau_\nu$$



[Malament \(2012\)](#)

[Schwartz \(2023\)](#)

[Schuller YT videos \(2015\)](#)



# A sidetone I can't resist: Newton-Cartan theory

A Newtonian space-time is the quadruplet  $(M, \tau, k, \nabla)$

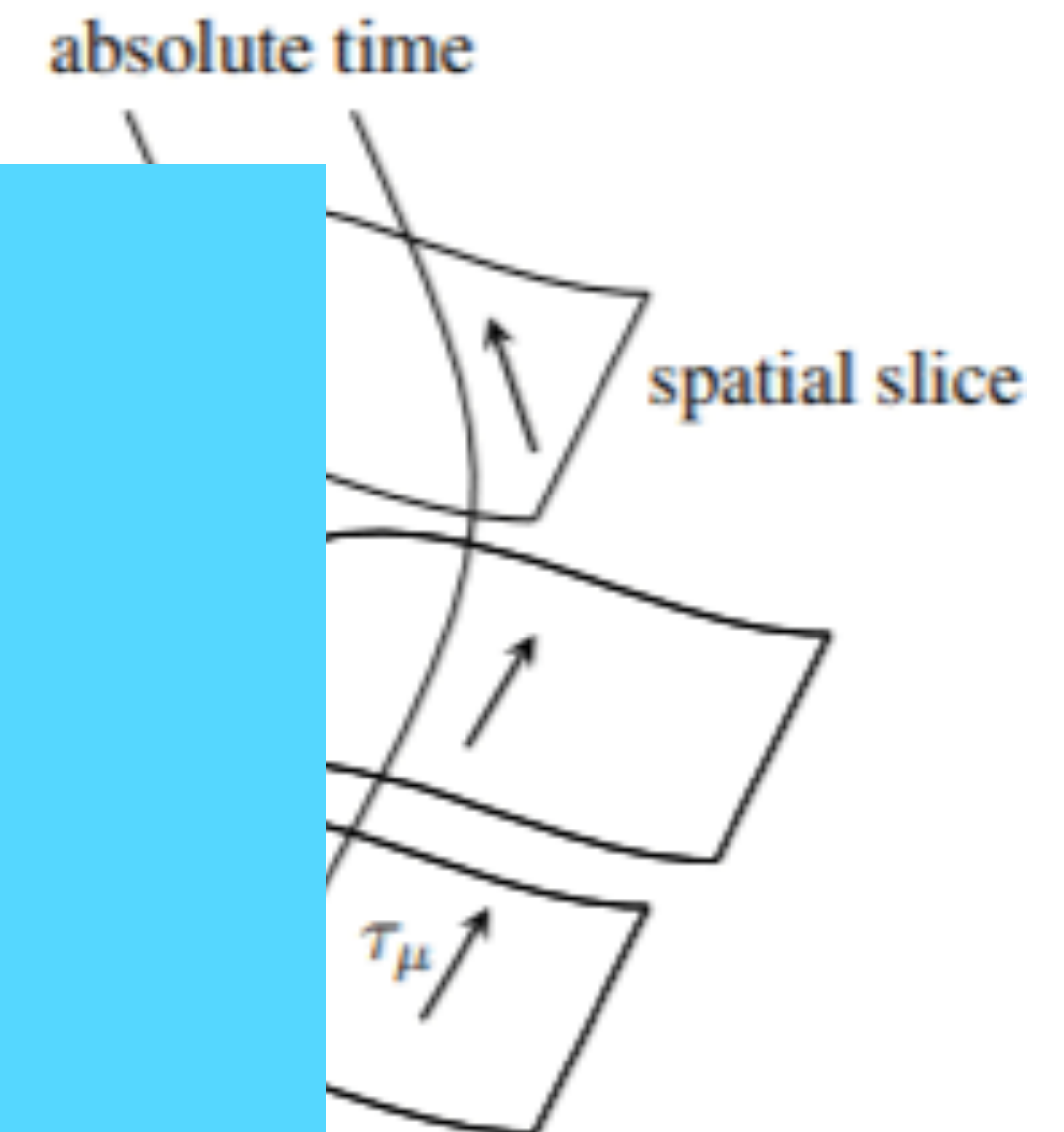
Here again, as everywhere in physics, **geometry is a powerful way to rewrite and reinterpret a theory**, showing other aspects:

- Thermodynamics (contact geometry)
- Quantum mechanics (Berry phase and geometric quantisation)
- Classical mechanics (symplectic geometry)
- Gauge theories (fiber bundles)
- ...

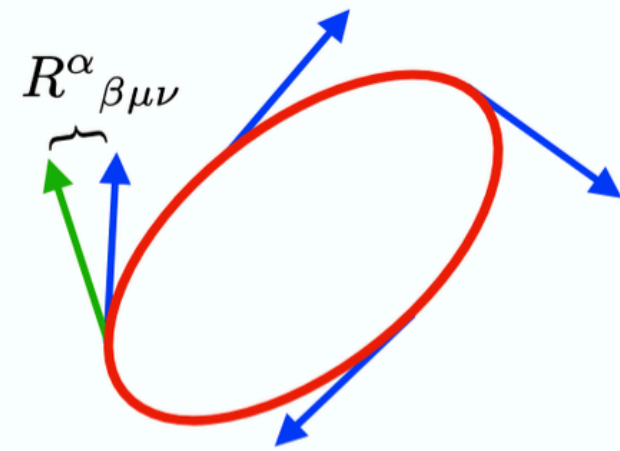
These are very difficult topics, but **SO BEAUTIFUL** and **very POWERFUL**

- Or in the single **Poisson equation**.

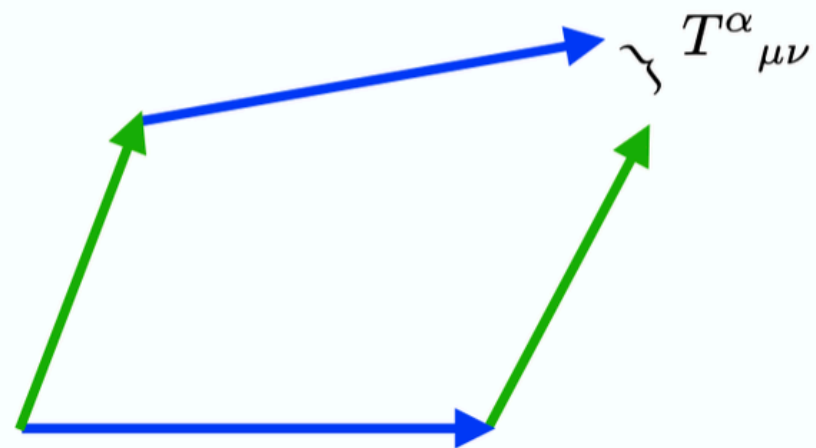
$$R_{\mu\nu} = 4\pi G\rho\tau_\mu\tau_\nu$$



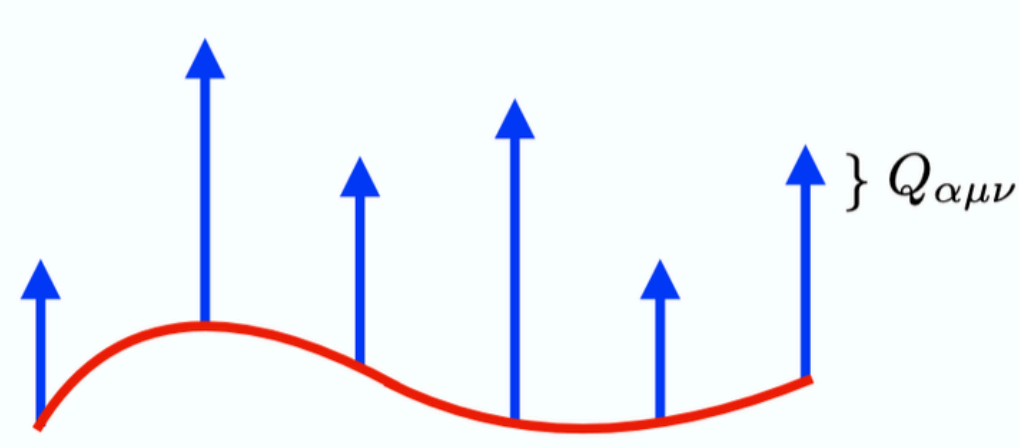
# The geometric trinity of gravity



The rotation of a vector transported along a closed curve is given by the curvature: General Relativity.



The non-closure of parallelograms formed when two vectors are transported along each other is given by the torsion: Teleparallel Equivalent of General Relativity.



The variation of the length of a vector as it is transported is given by the non-metricity: Symmetric Teleparallel Equivalent of General Relativity.

$$\Gamma^\rho_{\mu\nu} = \underbrace{\left\{ \begin{matrix} \rho \\ \mu\nu \end{matrix} \right\}}_{\text{Levi-Civita}} + \underbrace{K^\rho_{\mu\nu}}_{\text{contorsion}} + \underbrace{L^\rho_{\mu\nu}}_{\text{disformation}}$$

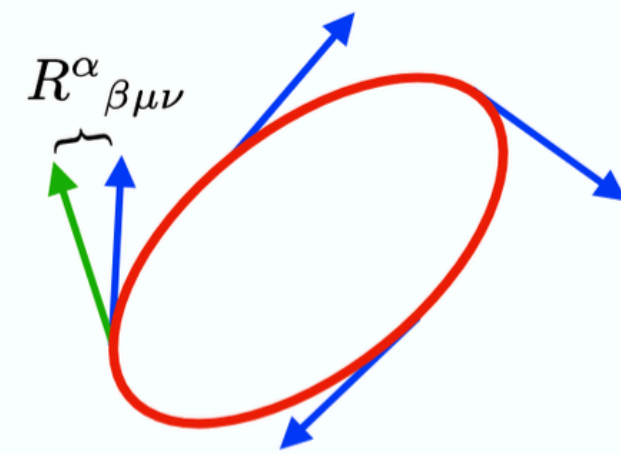
$$\left\{ \begin{matrix} \rho \\ \mu\nu \end{matrix} \right\} = \frac{1}{2} g^{\alpha\lambda} (\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})$$

$$K^\alpha_{\mu\nu} = \frac{1}{2} T^\alpha_{\mu\nu} + T_{(\mu}{}^\alpha{}_{\nu)}$$

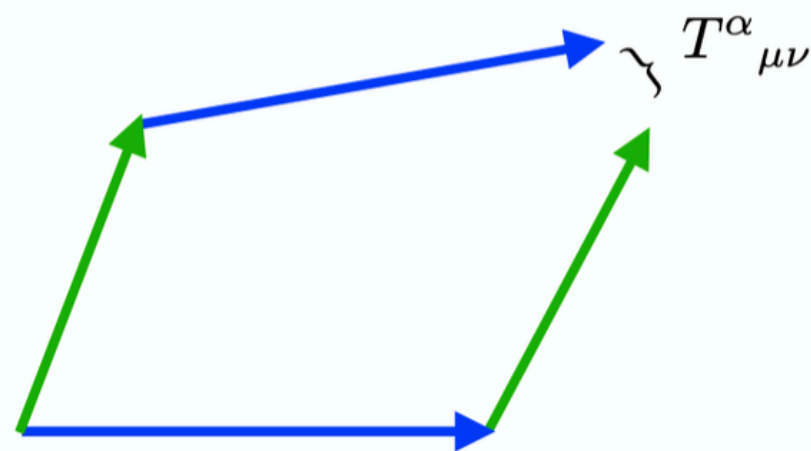
$$L^\alpha_{\mu\nu} = \frac{1}{2} Q^\alpha_{\mu\nu} - Q_{(\mu}{}^\alpha{}_{\nu)}$$



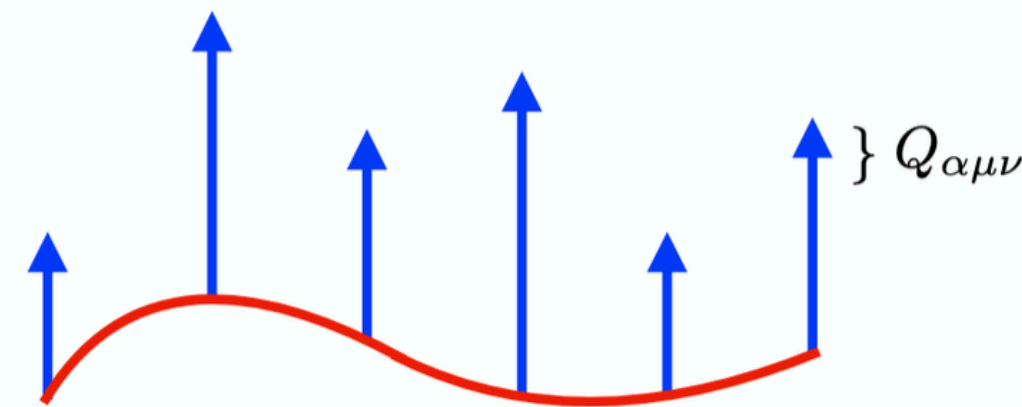
# The geometric trinity of gravity



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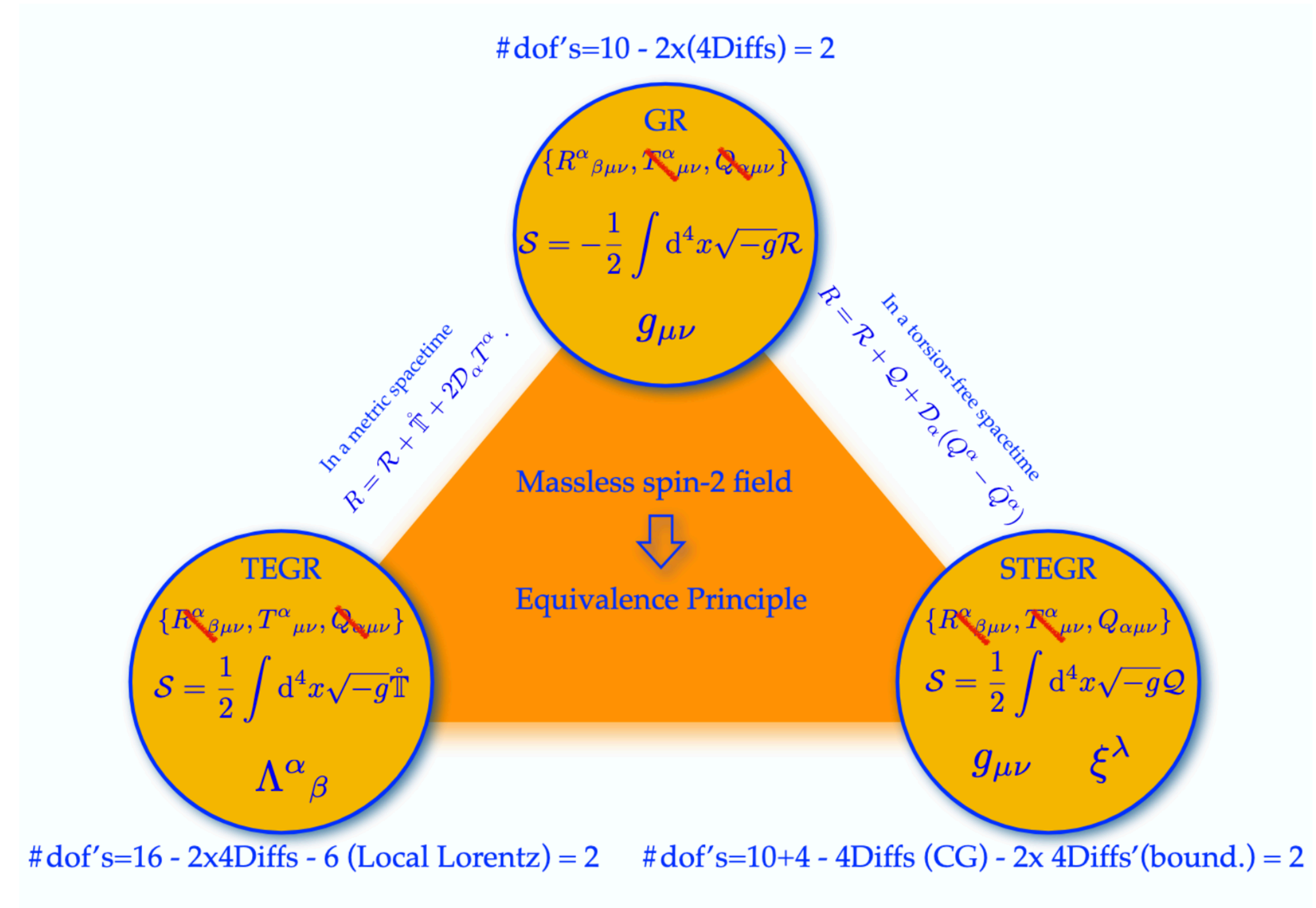


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Jiménez (2019)



Jiménez (2019)

# The geometric trinity of gravity

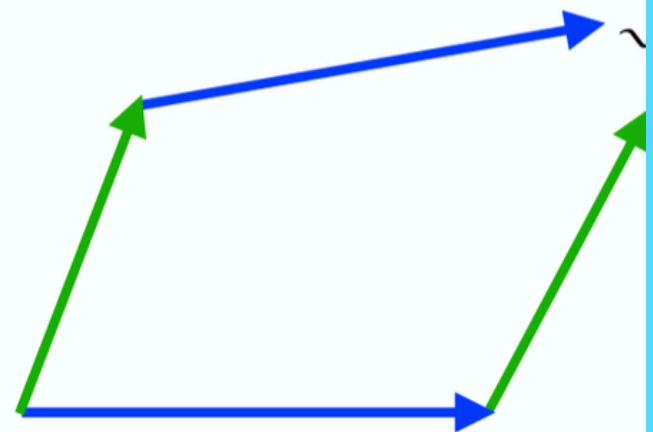
$$\# \text{dof's} = 10 - 2 \times (4 \text{Diffs}) = 2$$

Which is true? **Underdetermination of a theory** (classical philosophy of physics problem)

- Quantum theory (Copenhagen vs Bohemian pilot waves)
- AND SO MANY OTHERS

Keep this in mind when you tell your family that **«space-time is curved »**

The r  
curve



The non-closure of parallelograms formed when two vectors are transported along each other is given by the torsion: Teleparallel Equivalent of General Relativity.

The variation of the length of a vector as it is transported is given by the non-metricity: Symmetric Teleparallel Equivalent of General Relativity.

In a torsion-free spacetime  
 $R = \mathcal{R} + Q + D_\alpha(Q^\alpha - \tilde{Q}^\alpha)$

TEGR

$$\{R^\alpha_{\beta\mu\nu}, T^\alpha_{\mu\nu}, Q_{\alpha\mu\nu}\}$$

$$\mathcal{S} = \frac{1}{2} \int d^4x \sqrt{-g} \mathcal{Q}$$

$$g_{\mu\nu} \quad \xi^\lambda$$

$$\# \text{dof's} = 16 - 2 \times 4 \text{Diffs} - 6 (\text{Local Lorentz}) = 2 \quad \# \text{dof's} = 10 + 4 - 4 \text{Diffs (CG)} - 2 \times 4 \text{Diffs' (bound.)} = 2$$

Jiménez (2019)



# The geometric trinity of gravity

Higher orders modified gravity:

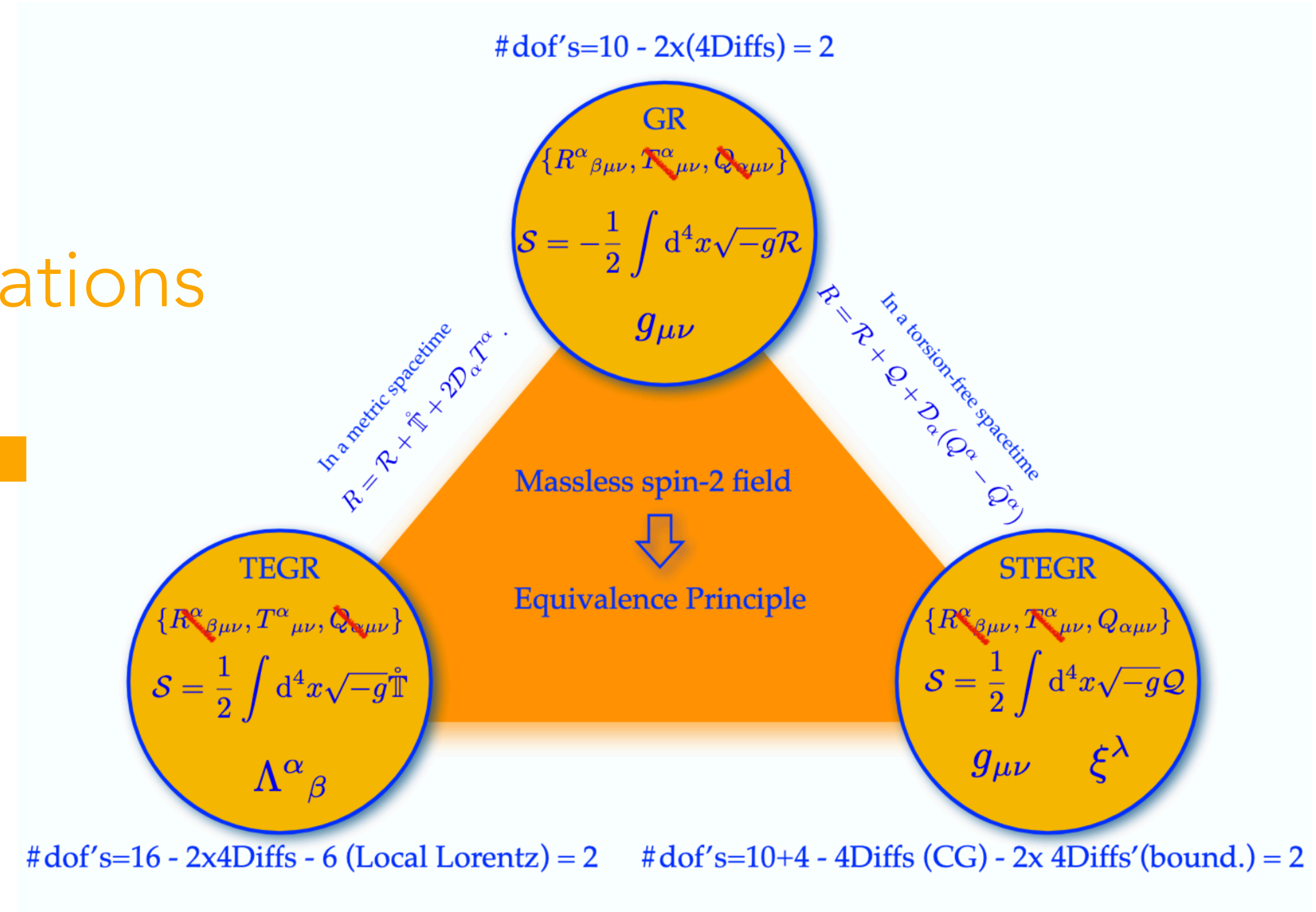
$$\int d^4x \sqrt{-g} \mathcal{R} + f(\mathcal{R})$$

$$-\int d^4x \sqrt{-g} \mathbb{T} + f(\mathbb{T})$$

$$-\int d^4x \sqrt{-g} \mathcal{Q} + f(\mathcal{Q})$$

All have widely different predictions!

Generalisations



Jiménez (2019)

# The geometric trinity of gravity

Higher orders modified gravity:

$$\int d^4x \sqrt{-g}$$

$$-\int d^4x \sqrt{-g}$$

$$-\int d^4x \sqrt{-g}$$

Conclusion of this loooong digression:

BY LOOKING AT **VARIOUS FACES OF GR**, WE GET DIFFERENT IDEAS ON **HOW TO EXTEND IT**

All have widely different predictions!

$$\# \text{dof's} = 10 - 2 \times (4 \text{Diffs}) = 2$$

In a torsion-free spacetime  
 $R = \mathcal{R} + Q + D_\alpha(Q^\alpha - \tilde{Q}^\alpha)$

TEGR

$$\{R^\alpha_{\beta\mu\nu}, T^\alpha_{\mu\nu}, Q_{\alpha\mu\nu}\}$$

$$\mathcal{S} = \frac{1}{2} \int d^4x \sqrt{-g} \mathcal{Q}$$

$$g_{\mu\nu} \quad \xi^\lambda$$

$$\# \text{dof's} = 16 - 2 \times 4 \text{Diffs} - 6 (\text{Local Lorentz}) = 2 \quad \# \text{dof's} = 10 + 4 - 4 \text{Diffs (CG)} - 2 \times 4 \text{Diffs' (bound.)} = 2$$

Jiménez (2019)



# Why going beyond GR?

- GR is an **INCREDIBLY successful** theory both **observationally** [lab, solar system, pulsars, cosmo..., weak, strong, extreme regimes (**quantum untouched**)] and **theoretically** (beautiful, robust, multiple faces)

## So why bother?

- **Why not?** We are physicists! (GR need to be tested against something + it is not perfect)
- **Dark energy and dark matter** (meh)
- **Cosmological tensions** (did DESI-DR2 really show a cross phantom divide?)
- **Quantum gravity** (implies higher order terms as quantum corrections!)

# How to go beyond GR? Lovelock

From geometry, Lovelock theorem (1971): The Einstein-Hilbert action with a cosmological constant is the only local, diffeomorphism-invariant action of the metric alone whose field equations are second order in 4D.

|    | Assumption                                                                                    | First class                                                                      |
|----|-----------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------|
| L1 | Four spacetime dimensions                                                                     | Hidden in S1                                                                     |
| L2 | The metric $g_{\mu\nu}$ is the only field                                                     | Assumption S2                                                                    |
| L3 | Field equations of at most second order in derivatives                                        | Hidden in D1                                                                     |
| L4 | Diffeomorphism invariance                                                                     | Hidden in the tensorial formalism (inherited from general covariance discussion) |
| L5 | Locality, and the equations follow from an action                                             | Hidden in D1 and D2                                                              |
| L6 | (implicit) The connection is the Levi-Civita connection of $g$ — no torsion, no non-metricity | S3                                                                               |

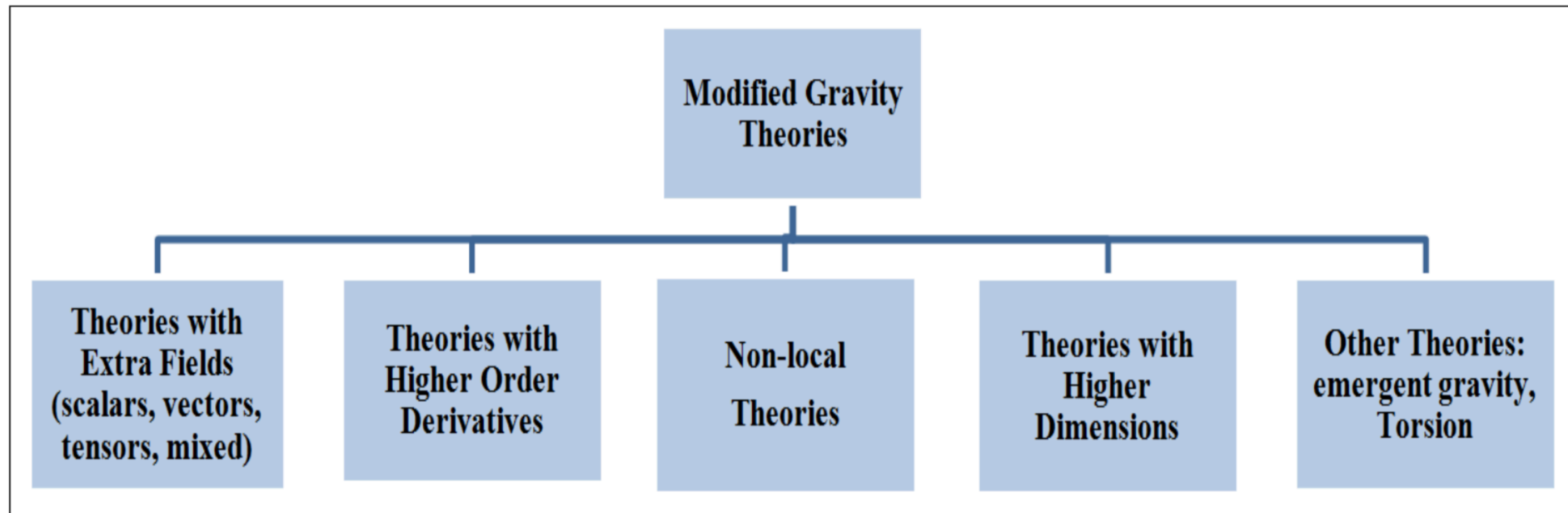


# How to go beyond GR? Weinberg-Deser

From field theory, Weinberg-Deser theorem (1964-1970): General relativity is the unique **consistent (no ghost)** low-energy theory of a **single self-interacting massless spin-2 field** on **flat four-dimensional** spacetime.

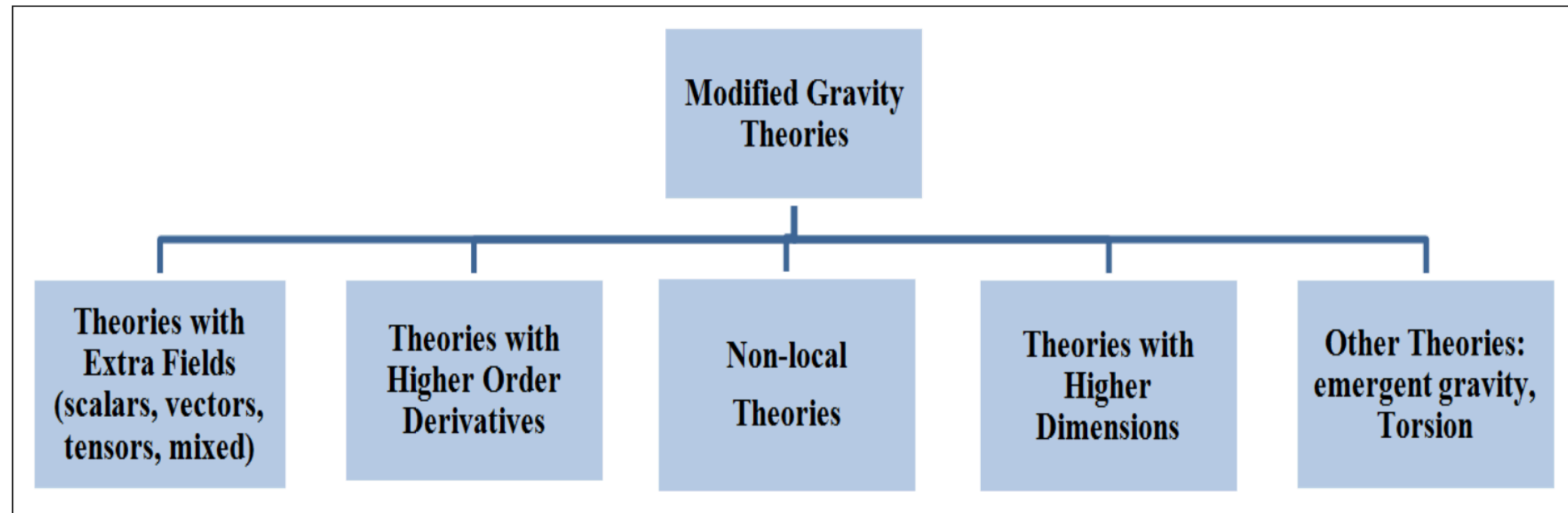
|    | Assumption                                                   |
|----|--------------------------------------------------------------|
| W1 | Exactly <b>one</b> spin-2 field                              |
| W2 | It is exactly <b>massless</b>                                |
| W3 | Poincaré (Lorentz) invariance                                |
| W4 | Unitarity / positive energy — no ghosts                      |
| W5 | Locality                                                     |
| W6 | Two-derivative, i.e. leading order in a derivative expansion |
| W7 | Four dimensions                                              |
| W8 | Flat (Minkowski) background                                  |
| W9 | No other light fields coupling to matter                     |

# How to go beyond GR?





# How to go beyond GR?

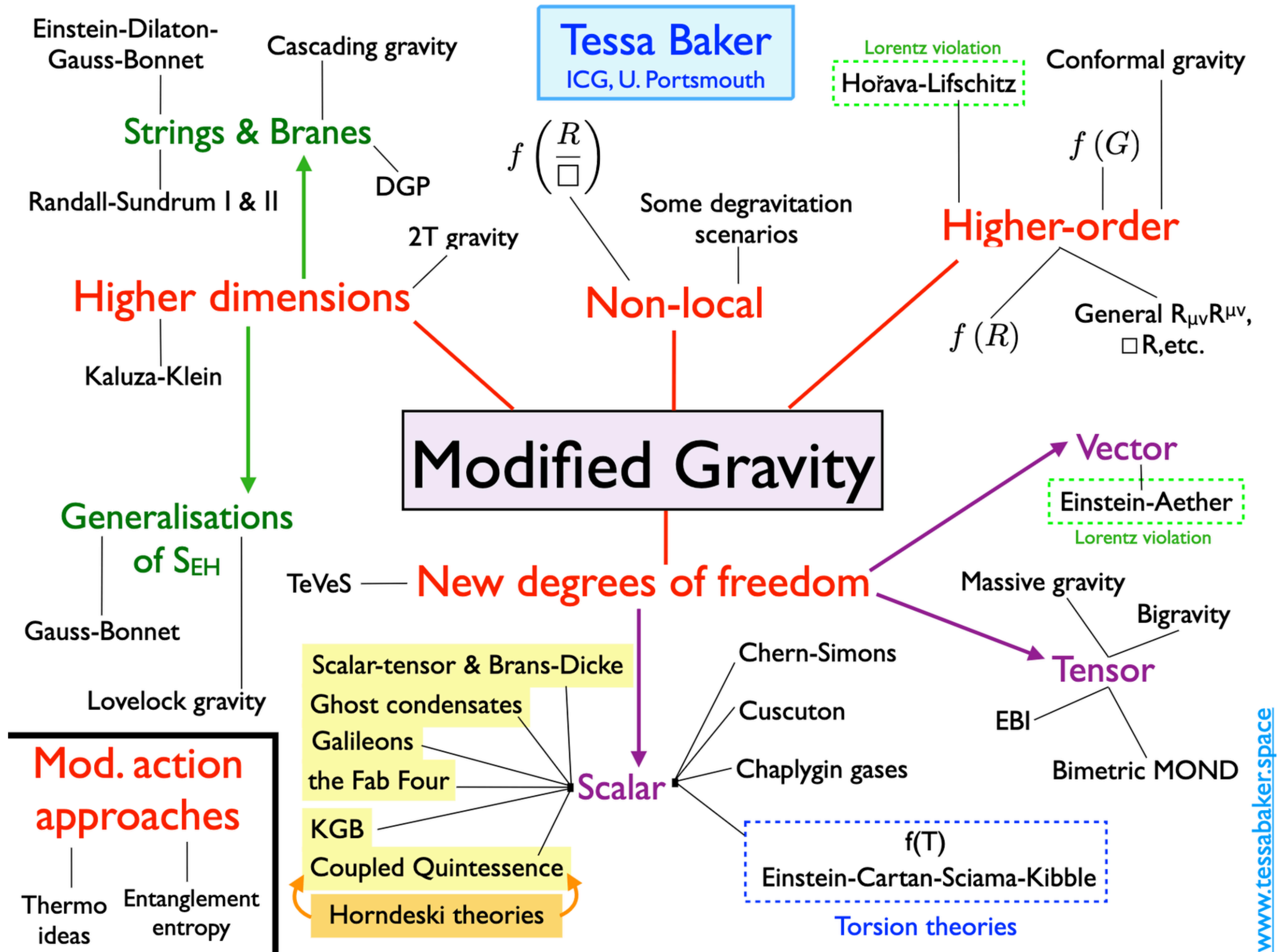


**QUANTUM GRAVITY REQUIRES THESE EXTENSIONS**

[Ishak \(2018\)](#)

**Tessa Baker**

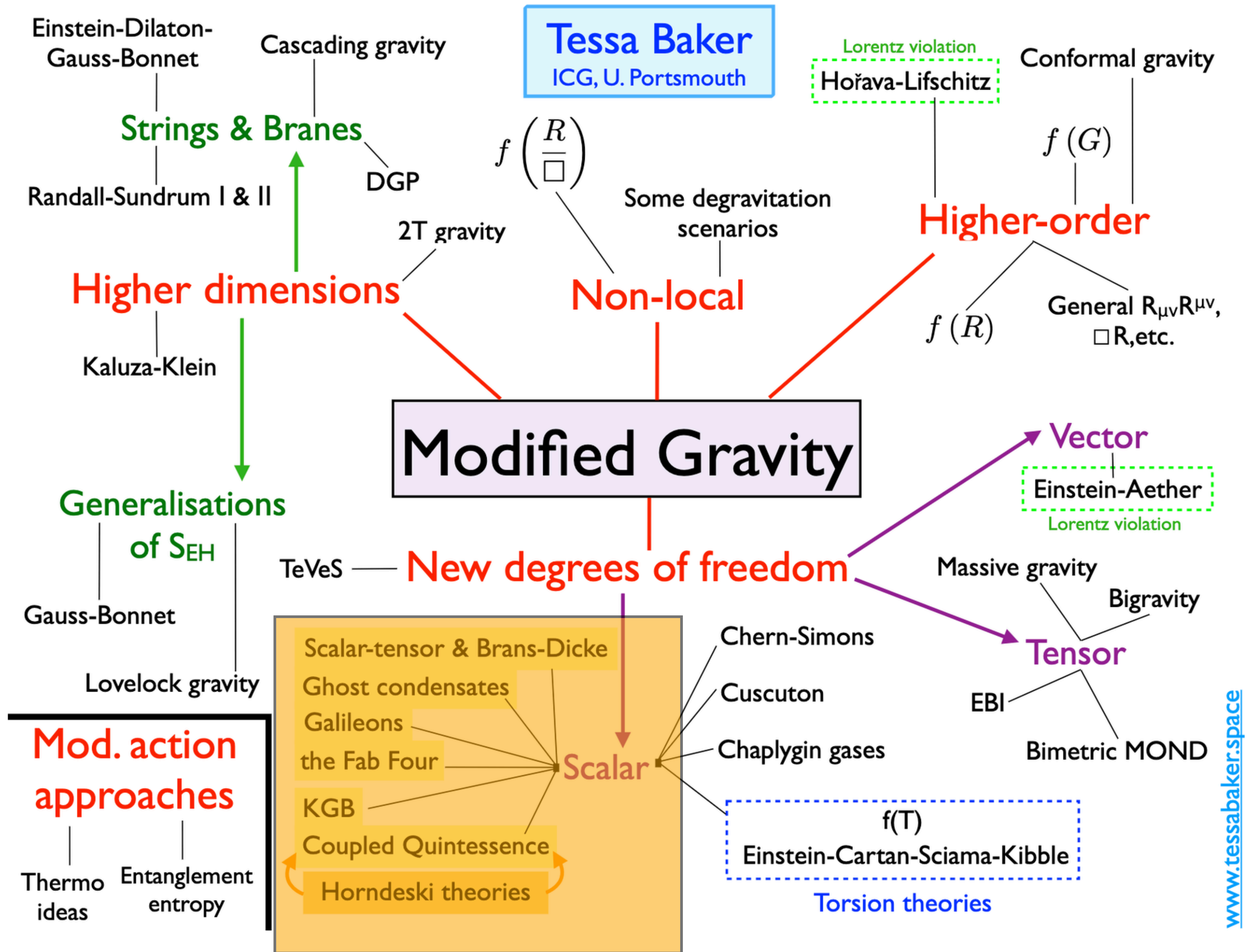
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**Tessa Baker**

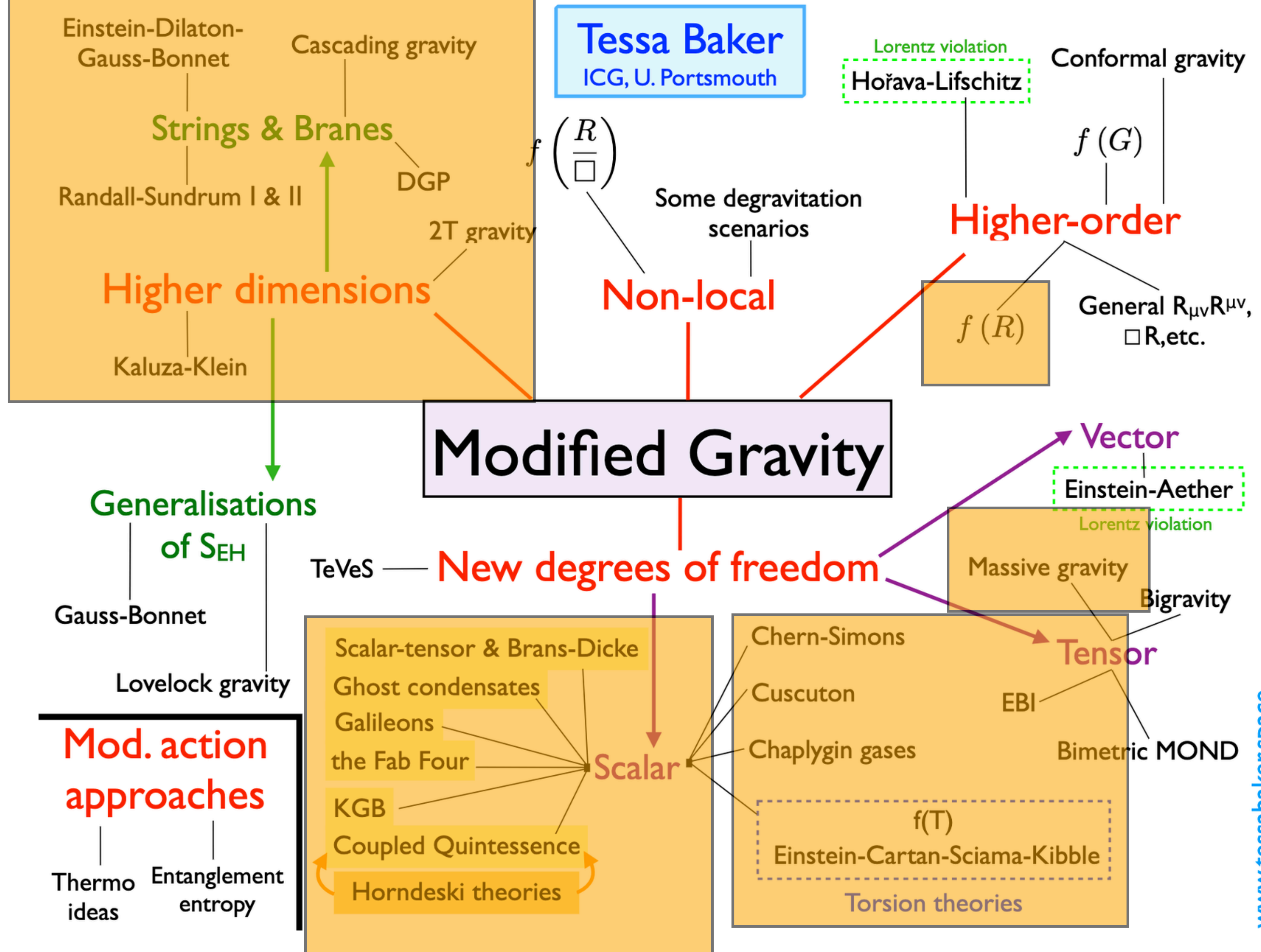
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Tessa Baker

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# Additional degrees of freedom: scalar fields

Cosmologists love scalar fields!

- **Easy to implement** without breaking Lorentz/diffeo invariance and other symmetries
- Nature approved: **Higgs** field !
- Solving **cosmological puzzles**: early symmetry breaking, **inflation**, (dynamical) dark energy ...
- **Theoretical necessity** in many models beyond the standard model (quantum gravity) and/or with extra-dimensions: Kaluza-Klein + **string theory**
- **Testable**: Clear stable predictions.
- **Higher-order generalizations** of GR  $\leftrightarrow$  Einstein-Hilbert action coupled to one or several scalar fields

# General considerations: Ghosts, instabilities and tachyons

At the perturbation level, one obtains **dispersion relations** like (or expanding action at second order):

$$\mathcal{F} \omega^2 = \mathcal{G} \frac{k^2}{a^2} + \mathcal{M}$$

$$\omega^2 = c_s^2 \frac{k^2}{a^2} + m^2$$

$$c_s^2 = \frac{\mathcal{G}}{\mathcal{F}}$$

$$m^2 = \frac{\mathcal{M}}{\mathcal{F}}$$

- If  $\mathcal{G} < 0$  it's a **ghost**  $\longrightarrow$  fatal for the theory
- If  $c_s^2 < 0$  it's a **gradient instability**  $\longrightarrow$  fatal for the theory
- If  $m^2 < 0$  it's a **tachyon**  $\longrightarrow$  not so great

It's a big challenge for modified gravity theory to avoid these guys!

The general methodology to look for them is spelled in the **notes**!





# General considerations: screening

There exist three types of screening to avoid **local bounds** while remaining cosmologically interesting:

$$\Phi$$

Chameleon

$$\nabla \Phi$$

K-mouflage

$$\nabla^2 \Phi$$

Vainshtein



No time for the details (See [P. Brax's](#) lecture)

Discussion: **Is it cheating?**

# General considerations: screening

There exist three types of screening to avoid **local bounds** while remaining cosmologically interesting:

$$\Phi$$

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No time for the details (See [P. Brax's](#) lecture)

Discussion: **Is it cheating?**

Could be, but **screening exists in nature!** (Electromagnetism, Higgs boson!)



# Scalar fields: Quintessence

A **scalar field** in curved space-time (**Quintessence**):

$$S = \int d^4x \sqrt{-|g|} \left( \frac{1}{16\pi G} (R - 2\Lambda) - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right)$$

**NOT MODIFIED GRAVITY!**

In **FLRW** space-time, equations of motion are **Klein-Gordon and Friedmann**:

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{\partial V}{\partial \phi} \qquad H^2 = \frac{8\pi G}{3} \left( \rho + \rho_\phi \right)$$

# Scalar fields: Brans-Dicke

A **scalar field** in curved space-time (**Brans-Dicke**):

$$S = \int d^4x \sqrt{-|g|} \left( \frac{1}{16\pi G} (\phi R - 2\Lambda) - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right)$$

## WHAT NOW?

Fantastic! This is our first proper modified gravity theory:

- Motivated from deep physical/philosophical principle: **Mach's principle** (see notes)
- **Varying  $G$ , or more precisely  $\phi \equiv 1/G(x, t)$** : violation of **SEP**, different metric for **gravity and matter**
- Must have cosmological **cool signatures**  $\mu \neq \Sigma \neq \eta \neq 1$



# Scalar fields: Brans-Dicke

A **scalar field** in curved space-time (**Brans-Dicke**): **Clean a bit the mess**

$$S = \int d^4x \sqrt{-|g|} \left( \frac{1}{16\pi} (\phi R - 2\Lambda) - \frac{\omega}{16\pi\phi} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right)$$

[\*Brans and Dicke \(1961\)\*](#)

The cosmological equations are already **more funky**! (with, historically  $V(\phi) = 0$ )

$$\ddot{\phi} + 3H\dot{\phi} = 8\pi \frac{\bar{\rho} - 3\bar{P}}{2\omega + 3}$$

**Klein-Gordon**

$$H^2 = \frac{8\pi\bar{\rho}}{3\phi} - \frac{K}{a^2} - H \frac{\dot{\phi}}{\phi} + \frac{\omega}{6} \frac{\dot{\phi}^2}{\phi^2}$$

**Friedmann**

# Scalar fields: Brans-Dicke

Under a suitable transformation (Einstein/Jordan frame)

$$S = \int d^4x \sqrt{-|g|} \left( \frac{1}{16\pi} (R[g] - 2\Lambda) - \frac{1}{2} g^{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} - V(\phi) \right) + \mathcal{L}_m(\psi, e^{-\kappa\tilde{\phi}} g_{\mu\nu})$$

$$\tilde{g}_{\mu\nu} = e^{-\kappa\tilde{\phi}} g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} \neq g_{\mu\nu}$$

**VIOLATION OF SEP**



# Scalar fields: Brans-Dicke

Under a suitable transformation (Einstein/Jordan frame)

$$S = \int d^4x \sqrt{-|g|} \left( \frac{1}{16\pi G} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} - V(\tilde{\phi}) \right) + \mathcal{L}_m(\psi, e^{-\kappa\tilde{\phi}} g_{\mu\nu})$$

Hides a lot of deep and complex features of modern field theories

CAN BE TREATED IN TUTORIAL CLASS

- Jordan vs Einstein frame
- Ambiguities on potential
- Cosmological evolution
- Constraints with data
- Screening

# General scalar-tensor theories

A **scalar field** in curved space-time (**scalar tensor**):

$$S = \int d^4x \sqrt{-|g|} \left( \frac{1}{16\pi} (F(\phi)R - 2\Lambda) - K(\phi, \partial_\mu \phi) \partial^\mu \phi - V(\phi) \right)$$



Generalized BD



k-essence



Potential



# **$f(R)$ is a general scalar-tensor theory**

More complex functions of the Ricci scalar

$$S = \int d^4x \sqrt{-|g|} \frac{1}{16\pi} R \quad \longrightarrow \quad S = \int d^4x \sqrt{-|g|} \frac{1}{16\pi} f(R)$$

Many different functions can be used, providing models for **dark energy and inflation**

# $f(R)$ is a general scalar-tensor theory

More complex functions of the Ricci scalar

$$f_R = \frac{df(R)}{dR}$$

$$S = \int d^4x \sqrt{-|g|} \frac{1}{16\pi} f(R) = \int d^4x \sqrt{-|g|} \frac{1}{16\pi} (f_R R - [R f_R - f(R)])$$

  $\phi \equiv f_R$

$$= \int d^4x \sqrt{-|g|} \frac{1}{16\pi} (\underbrace{\phi R}_{F(\phi)R} - \underbrace{[R\phi - f(R)]}_{V(\phi)})$$

$K(\phi) = 0$

This is Brans-Dicke theory with  $\omega = 0$  and potential  $V(\phi) = R\phi - f(R)$

The scalar field is called **the scalaron**

Going to Einstein frame, we see that scalaron is screened as a Chameleon with  $\beta = -1/\sqrt{6}$



# $f(R)$ is a general scalar-tensor theory

More complex functions of the Ricci scalar

$$S = \int d^4x \sqrt{-|g|} \frac{1}{16\pi} f(R)$$

$$f_R = \frac{df(R)}{dR}$$

$$f(R) = R + \frac{R^2}{6M^2}$$

Starobinski inflation!

[Starobinski \(1980\)](#)

$$f(R) = R - M^2 \frac{c_1(R/M^2)^n}{c_2(R/M^2)^n + 1} \simeq R - \frac{c_1}{c_2} M^2 + \frac{c_1}{c_2^2} M^2 \left( \frac{M^2}{R} \right)^n$$

Cosmic acceleration!

[Hu-Sawicki+ \(2007\)](#)

$$M^2 = H_0^2 \Omega_m$$

# $f(R)$ is a general scalar-tensor theory

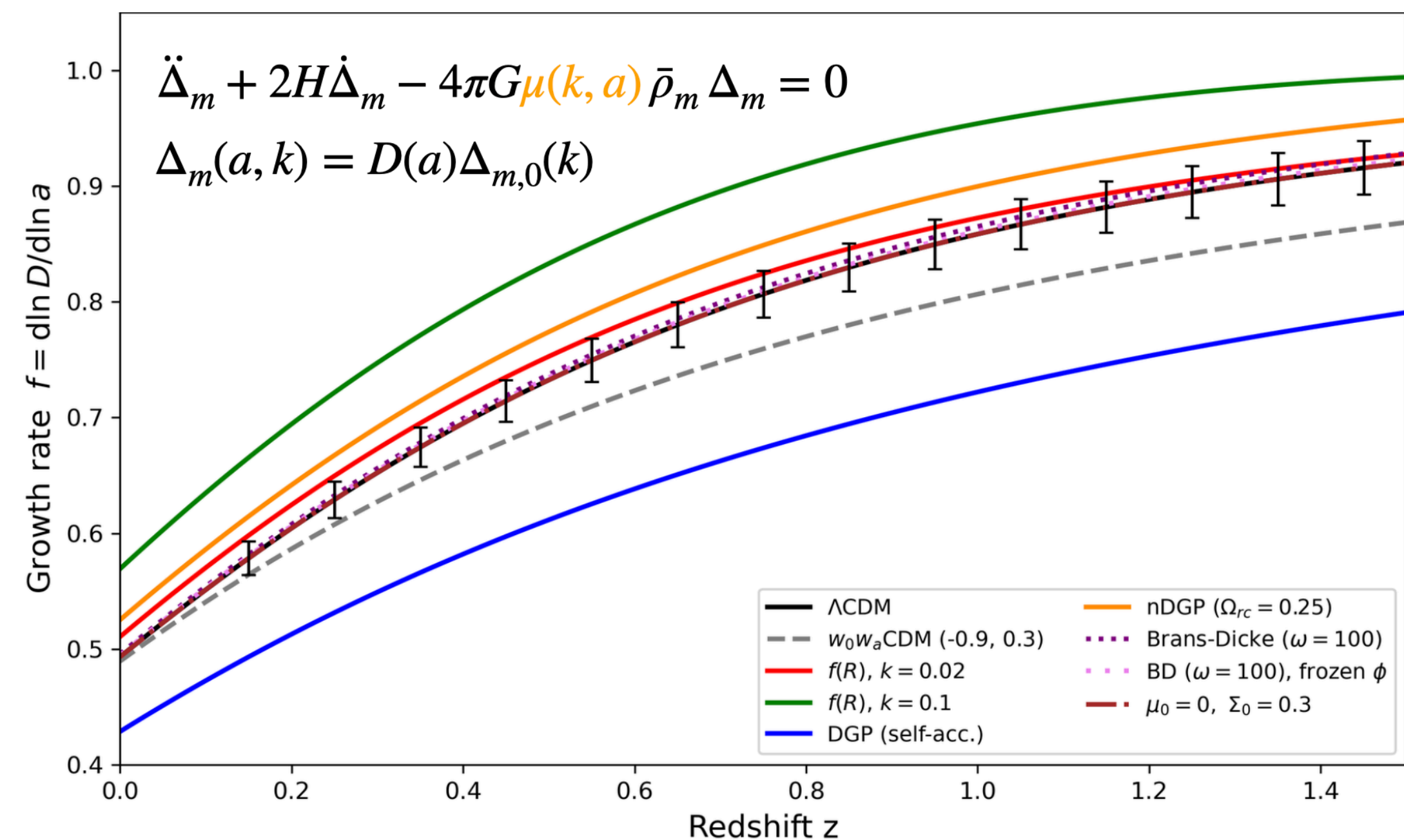
More complex functions of the Ricci scalar

$$S = \int d^4x \sqrt{-|g|} \frac{1}{16\pi} f(R)$$

$$f_R = \frac{df(R)}{dR}$$

$$\mu(a, k) = \frac{1}{f_R} \frac{4 + 3a^2 m^2 / k^2}{3 + 3a^2 m^2 / k^2},$$

$$\Sigma(a, k) = \frac{1}{f_R}$$





# Even more general scalar fields: Horndeski

The **most general second order** containing a scalar field and a metric:

$$S_H = \int d^4x \sqrt{-g} (\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5) + S_m[g_{\mu\nu}, \psi_m]$$



$$\mathcal{L}_2 = G_2(\phi, X)$$

$$\mathcal{L}_3 = -G_3(\phi, X) \square\phi$$

$$\mathcal{L}_4 = G_4(\phi, X) R + \frac{dG_4}{dX} \left[ (\square\phi)^2 - \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi \right]$$

$$\mathcal{L}_5 = G_5(\phi, X) G_{\mu\nu} \nabla^\mu \nabla^\nu \phi - \frac{1}{6} \frac{dG_5}{dX} \left[ (\square\phi)^3 - 3 \square\phi \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi + 2(\nabla_\mu \nabla_\nu \phi \nabla^\nu \nabla^\rho \phi \nabla_\rho \nabla^\mu \phi) \right]$$

[Horndeski \(1974\)](#)

Second order is to avoid ghosts. Higher orders with no ghost are called « **DHOST** » theories [Langlois \(2017\)](#)

# Even more general scalar fields: Horndeski

The **most general second order** containing a scalar field and a metric:

$$S_H = \int d^4x \sqrt{-g} (\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5) + S_m[g_{\mu\nu}, \psi_m]$$



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$$\mathcal{L}_5 = G_5(\phi, X) G_{\mu\nu} \nabla^\mu \nabla^\nu \phi - \frac{1}{6} \frac{dG_5}{dX} \left[ (\square\phi)^3 - 3 \square\phi \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi + 2(\nabla_\mu \nabla_\nu \phi \nabla^\nu \nabla^\rho \phi \nabla_\rho \nabla^\mu \phi) \right]$$

**Kinetic terms, sometimes noted K. Contains full quintessence Lagrangian (including possible potential)**



# Even more general scalar fields: Horndeski

The **most general second order** containing a scalar field and a metric:

$$S_H = \int d^4x \sqrt{-g} (\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5) + S_m[g_{\mu\nu}, \psi_m]$$



$$\mathcal{L}_2 = G_2(\phi, X)$$

$$\mathcal{L}_3 = -G_3(\phi, X) \square\phi$$

Possible screening here

$$\mathcal{L}_4 = G_4(\phi, X) R + \frac{dG_4}{dX} \left[ (\square\phi)^2 - \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi \right]$$

$$\mathcal{L}_5 = G_5(\phi, X) G_{\mu\nu} \nabla^\mu \nabla^\nu \phi - \frac{1}{6} \frac{dG_5}{dX} \left[ (\square\phi)^3 - 3 \square\phi \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi + 2(\nabla_\mu \nabla_\nu \phi \nabla^\nu \nabla^\rho \phi \nabla_\rho \nabla^\mu \phi) \right]$$

# Even more general scalar fields: Horndeski

The **most general second order** containing a scalar field and a metric:

$$S_H = \int d^4x \sqrt{-g} (\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5) + S_m[g_{\mu\nu}, \psi_m]$$



$$\mathcal{L}_2 = G_2(\phi, X)$$

$$\mathcal{L}_3 = -G_3(\phi, X) \square\phi$$

**GR is here + possible Brans-Dicke modification**

$$\mathcal{L}_4 = \boxed{G_4(\phi, X) R} + \frac{dG_4}{dX} \left[ (\square\phi)^2 - \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi \right]$$

$$\mathcal{L}_5 = G_5(\phi, X) G_{\mu\nu} \nabla^\mu \nabla^\nu \phi - \frac{1}{6} \frac{dG_5}{dX} \left[ (\square\phi)^3 - 3 \square\phi \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi + 2(\nabla_\mu \nabla_\nu \phi \nabla^\nu \nabla^\rho \phi \nabla_\rho \nabla^\mu \phi) \right]$$



# Even more general scalar fields: Horndeski

The **most general second order** containing a scalar field and a metric:

$$S_H = \int d^4x \sqrt{-g} (\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5) + S_m[g_{\mu\nu}, \psi_m]$$



$$\mathcal{L}_2 = G_2(\phi, X)$$

$$\mathcal{L}_3 = -G_3(\phi, X) \square\phi$$

Crazy non linear stuffs!

$$\mathcal{L}_4 = G_4(\phi, X) R + \frac{dG_4}{dX} \left[ (\square\phi)^2 - \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi \right]$$

$$\mathcal{L}_5 = G_5(\phi, X) G_{\mu\nu} \nabla^\mu \nabla^\nu \phi - \frac{1}{6} \frac{dG_5}{dX} \left[ (\square\phi)^3 - 3 \square\phi \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi + 2(\nabla_\mu \nabla_\nu \phi \nabla^\nu \nabla^\rho \phi \nabla_\rho \nabla^\mu \phi) \right]$$

# Even more general scalar fields: Horndeski

| Theory                              | $G_2$                                                 | $G_3$                     | $G_4$                                      | $G_5$              |
|-------------------------------------|-------------------------------------------------------|---------------------------|--------------------------------------------|--------------------|
| GR                                  | 0                                                     | 0                         | $M_{\text{Pl}}^2/2$                        | 0                  |
| GR + $\Lambda$                      | $-M_{\text{Pl}}^2\Lambda$                             | 0                         | $M_{\text{Pl}}^2/2$                        | 0                  |
| Quintessence                        | $X - V(\phi)$                                         | 0                         | $M_{\text{Pl}}^2/2$                        | 0                  |
| Brans–Dicke                         | $2\omega X/\phi$                                      | 0                         | $\phi/16\pi$                               | 0                  |
| k-essence                           | $G_2(\phi, X)$                                        | 0                         | $M_{\text{Pl}}^2/2$                        | 0                  |
| $f(R)$                              | $-\frac{M_{\text{Pl}}^2}{2}(\phi f'(\phi) - f(\phi))$ | 0                         | $\frac{M_{\text{Pl}}^2}{2}f'(\phi)$        | 0                  |
| Kinetic gravity braiding            | $G_2(\phi, X)$                                        | $G_3(\phi, X)$            | $M_{\text{Pl}}^2/2$                        | 0                  |
| Cubic Galileon                      | $X$                                                   | $c_3 X$                   | $M_{\text{Pl}}^2/2$                        | 0                  |
| Non-minimal $\xi\phi^2 R$           | $X - V$                                               | 0                         | $\frac{1}{2}(M_{\text{Pl}}^2 + \xi\phi^2)$ | 0                  |
| Gauss–Bonnet $\xi(\phi)\mathcal{G}$ | $8\xi^{(4)}X^2(3 - \ln X)$                            | $4\xi^{(3)}X(7 - 3\ln X)$ | $4\xi^{(2)}X(2 - \ln X)$                   | $-4\xi^{(1)}\ln X$ |



# Even more general scalar fields: Horndeski

Good news: All the **perturbation theory** of Horndeski theories can be summarised in four functions

$\alpha_M, \alpha_T, \alpha_B, \alpha_K$

[Bellini and Sawicki \(2014\)](#)

| function                    | name                        | meaning                                                                                                                        |
|-----------------------------|-----------------------------|--------------------------------------------------------------------------------------------------------------------------------|
| $\alpha_M \equiv \nu_T - 3$ | <b>Planck-mass run rate</b> | rate of change of the effective gravitational constant or Planck mass, related to the friction $\nu_T$ of gravitational waves. |
| $\alpha_T \equiv c_T^2 - 1$ | <b>tensor speed excess</b>  | gravitational waves travel at $c_T \neq c$                                                                                     |
| $\alpha_B$                  | <b>braiding</b>             | kinetic mixing between the scalar and the metric; makes dark energy cluster                                                    |
| $\alpha_K$                  | <b>kineticity</b>           | the scalar's own kinetic energy; controls its sound speed                                                                      |

$$M_*^2 = 2 \left( G_4 - 2X \frac{dG_4}{dX} + X \frac{dG_5}{d\phi} - XH \frac{d\phi}{dt} \frac{dG_5}{dX} \right)$$

$$\begin{aligned} \alpha_M &\equiv \frac{d \ln M_*^2}{d \ln a} \\ &= \frac{2}{M_*^2} \left[ 2X \left( \frac{dG_4}{dX} - \frac{dG_5}{d\phi} + 2X \left( \frac{d^2 G_4}{dX^2} - \frac{d^2 G_5}{dX d\phi} \right) \right) \right. \\ &\quad + \frac{1}{H} \frac{d\phi}{dt} \left( \frac{dG_4}{d\phi} - X \left( 2 \frac{d^2 G_4}{dX d\phi} - \frac{d^2 G_5}{d\phi^2} \right) \right) + XH \frac{d\phi}{dt} \left( 3 \frac{dG_5}{dX} + 2X \frac{d^2 G_5}{dX^2} \right) \\ &\quad - \frac{X}{H} \frac{dG_5}{dX} \frac{dH}{dt} \frac{d\phi}{dt} \\ &\quad \left. - \left( 3X \frac{dG_5}{dX} + 2X^2 \frac{d^2 G_5}{dX^2} + \frac{1}{H} \frac{d\phi}{dt} \left( \frac{dG_4}{dX} - \frac{dG_5}{d\phi} + X \left( 2 \frac{d^2 G_4}{dX^2} - \frac{d^2 G_5}{dX d\phi} \right) \right) \right) \left( \frac{d^2 \phi}{dt^2} + H \frac{d\phi}{dt} \right) \right] \end{aligned}$$

$$\begin{aligned} \alpha_K &= \frac{2X}{H^2 M_*^2} \left[ \frac{dG_2}{dX} - 2 \frac{dG_3}{d\phi} + 2X \left( \frac{d^2 G_2}{dX^2} - \frac{d^2 G_3}{dX d\phi} \right) \right. \\ &\quad + 6H \frac{d\phi}{dt} \left( \frac{dG_3}{dX} - 3 \frac{d^2 G_4}{dX d\phi} + X \left( \frac{d^2 G_3}{dX^2} - 2 \frac{d^3 G_4}{dX^2 d\phi} \right) \right) \\ &\quad + 6H^2 \left( \frac{dG_4}{dX} - \frac{dG_5}{d\phi} + X \left( 8 \frac{d^2 G_4}{dX^2} - 5 \frac{d^2 G_5}{dX d\phi} \right) + 2X^2 \left( 2 \frac{d^3 G_4}{dX^3} - \frac{d^3 G_5}{dX^2 d\phi} \right) \right) \\ &\quad \left. + 2H^3 \frac{d\phi}{dt} \left( 3 \frac{dG_5}{dX} + 7X \frac{d^2 G_5}{dX^2} + 2X^2 \frac{d^3 G_5}{dX^3} \right) \right] \end{aligned}$$

$$\begin{aligned} \alpha_B &= \frac{2}{HM_*^2} \frac{d\phi}{dt} \left[ - \frac{dG_4}{d\phi} + X \left( \frac{dG_3}{dX} - 2 \frac{d^2 G_4}{dX d\phi} \right) \right. \\ &\quad \left. + 2H \frac{d\phi}{dt} \left( \frac{dG_4}{dX} - \frac{dG_5}{d\phi} + X \left( 2 \frac{d^2 G_4}{dX^2} - \frac{d^2 G_5}{dX d\phi} \right) \right) + XH^2 \left( 3 \frac{dG_5}{dX} + 2X \frac{d^2 G_5}{dX^2} \right) \right] \end{aligned}$$

$$\alpha_T \equiv c_T^2 - 1 = \frac{2X}{M_*^2} \left[ 2 \frac{dG_4}{dX} - 2 \frac{dG_5}{d\phi} - \left( \frac{d^2 \phi}{dt^2} - H \frac{d\phi}{dt} \right) \frac{dG_5}{dX} \right]$$



# The devastating constraints of gravitational waves

Tensor perturbations:

$$\ddot{h}_{ij} + \underbrace{(3 + \alpha_M)}_{\nu_T} H \dot{h}_{ij} + \underbrace{(1 + \alpha_T)}_{c_T^2} \frac{k^2}{a^2} h_{ij} = 0$$

Remember that bounds on  $|c_T - c|$  are super-tight, they lead to  $|\alpha_T| \lesssim 10^{-15}$ . This can safely be assumed as null.

All these equations can be proven in tutorial

Propagating this bound, we get  $G_5 = 0$  and  $dG_4/dX = 0$ , simplifying drastically the Lagrangian to:

$$\mathcal{L} = G_2(\phi, X) - G_3(\phi, X) \square \phi + G_4(\phi) R$$

Quintessence/k-essence

Screening

Brans-Dicke

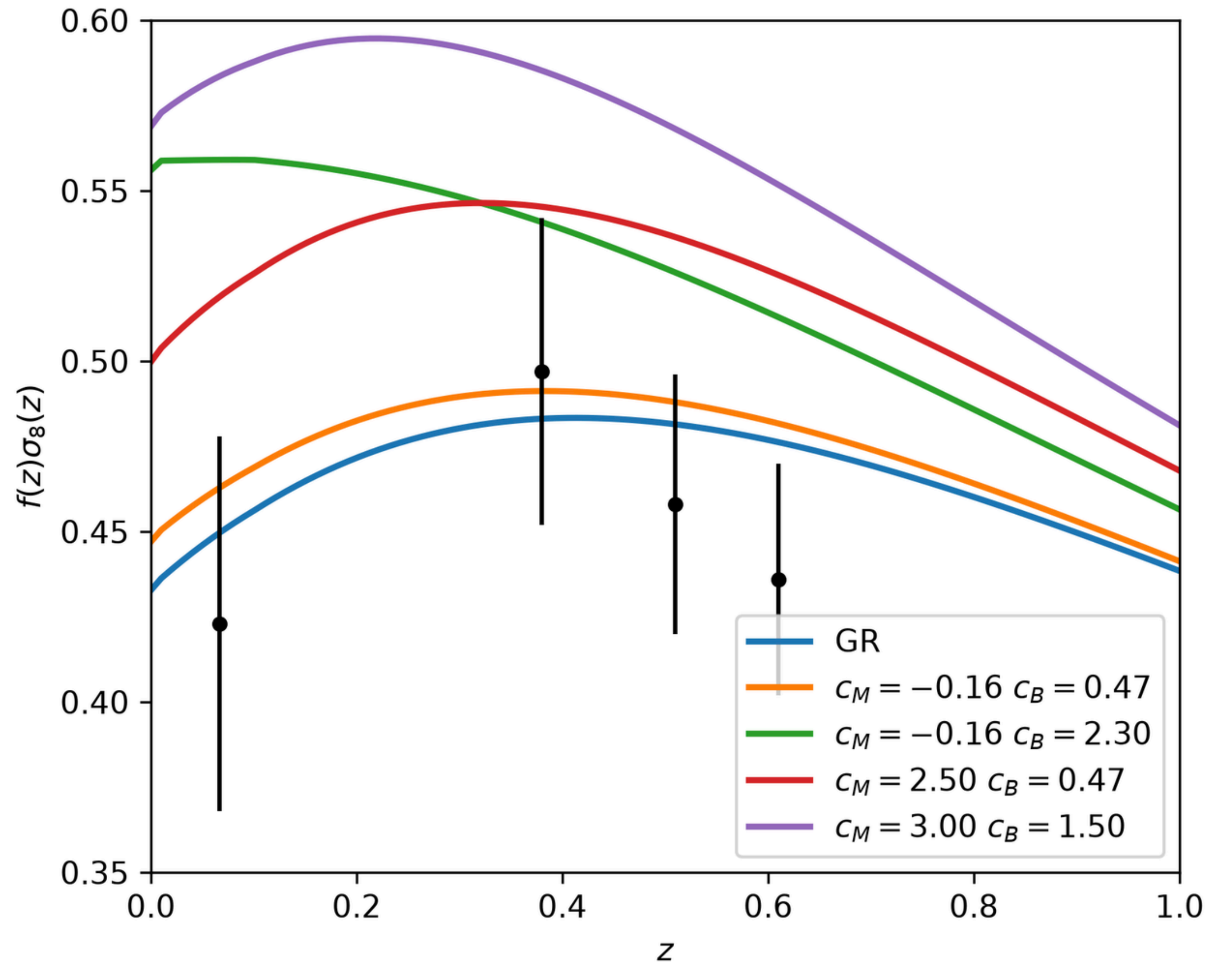
We talk about **Luminal Horndeski theories**

# The devastating constraints of gravitational waves

In luminal Horndeski,  
**scalar perturbations** lead to:

$$\mu(z) = \frac{1}{8\pi G M_*^2} \left( 1 + \frac{2(\alpha_M + \alpha_B/2)^2}{c_s^2(\alpha_K + 3\alpha_B^2/2)} \right)$$

$$\Sigma(z) = \frac{1}{8\pi G M_*^2} \left( 1 + \frac{(\alpha_M + \frac{\alpha_B}{2})(2\alpha_M + 2\alpha_B)}{c_s^2(\alpha_K + \frac{3}{2}\alpha_B^2)} \right)$$

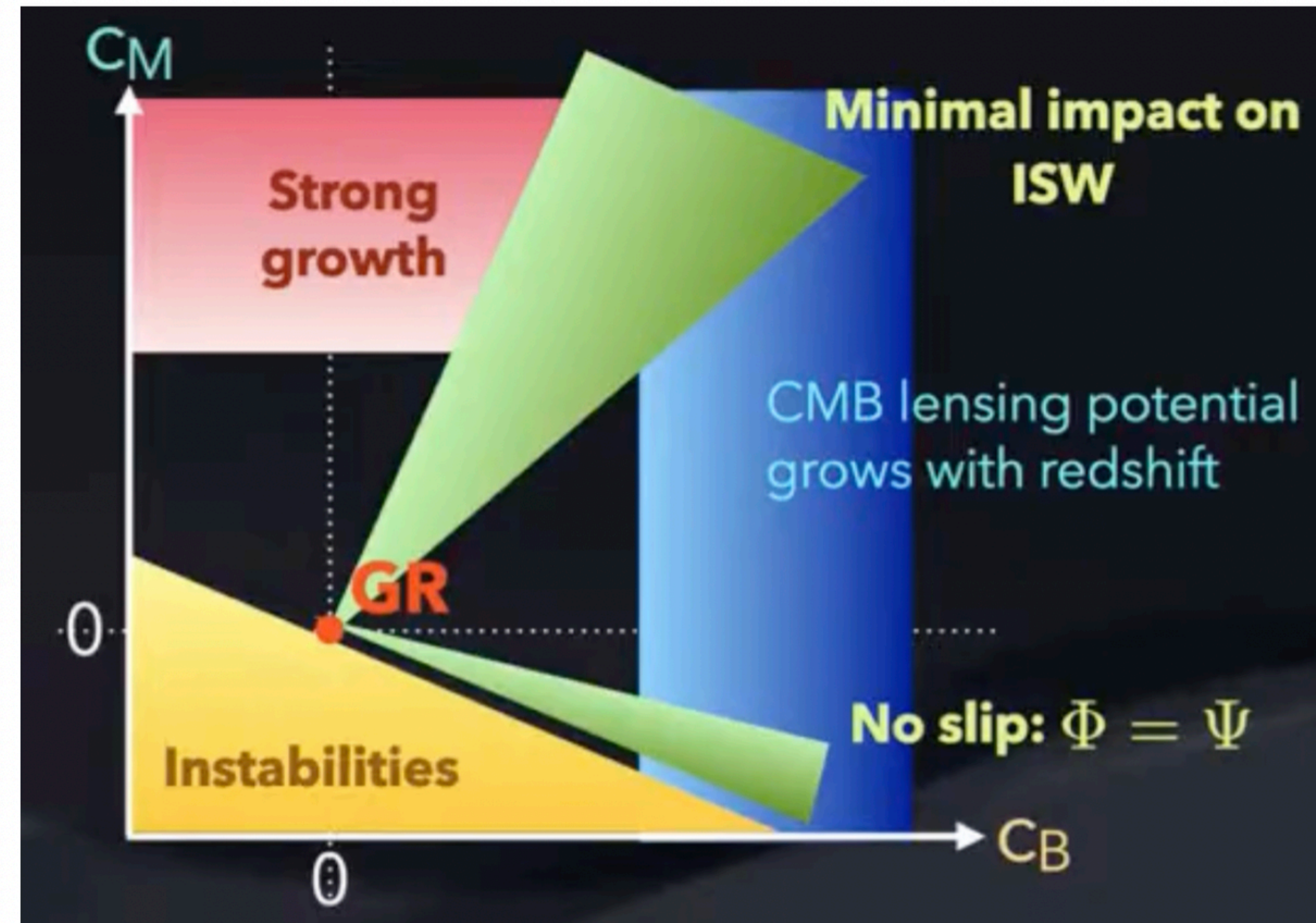
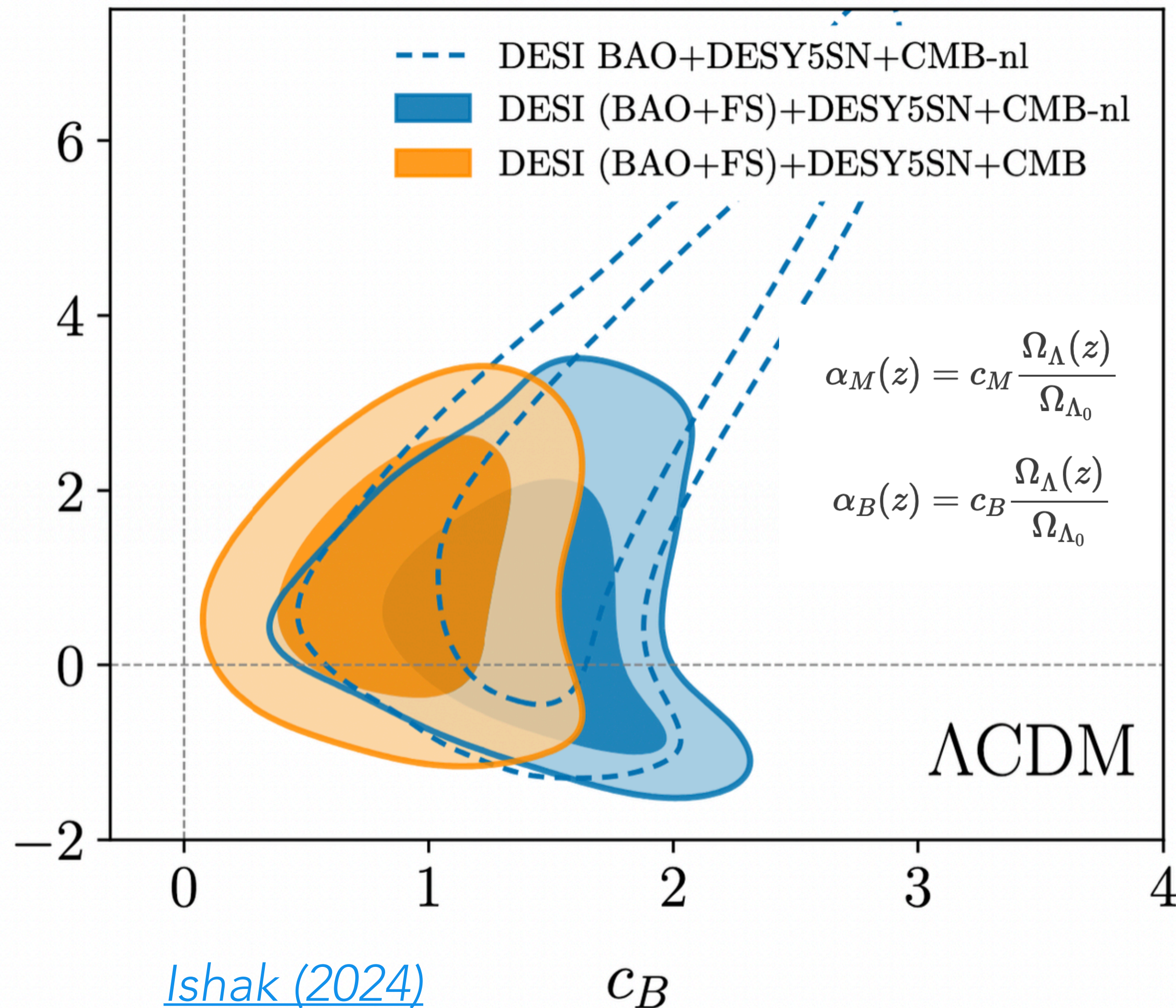


Plot made with: [https://github.com/mcataneo/mochi\\_class\\_public](https://github.com/mcataneo/mochi_class_public)



# Scalar fields: Horndeski

Constraints from BAO + CMB + LSS-FS



*T. Baker lectures*



# Beyond Horndeski

- Higher spin: Proca, TeVs
- Higher orders :  $f(R)$ ,  $f(T)$ ,  $f(Q)$ ,  $f(G)$  ...
- More dimensions : DPG gravity
- Thermodynamic gravity
- Non local gravity
- ...

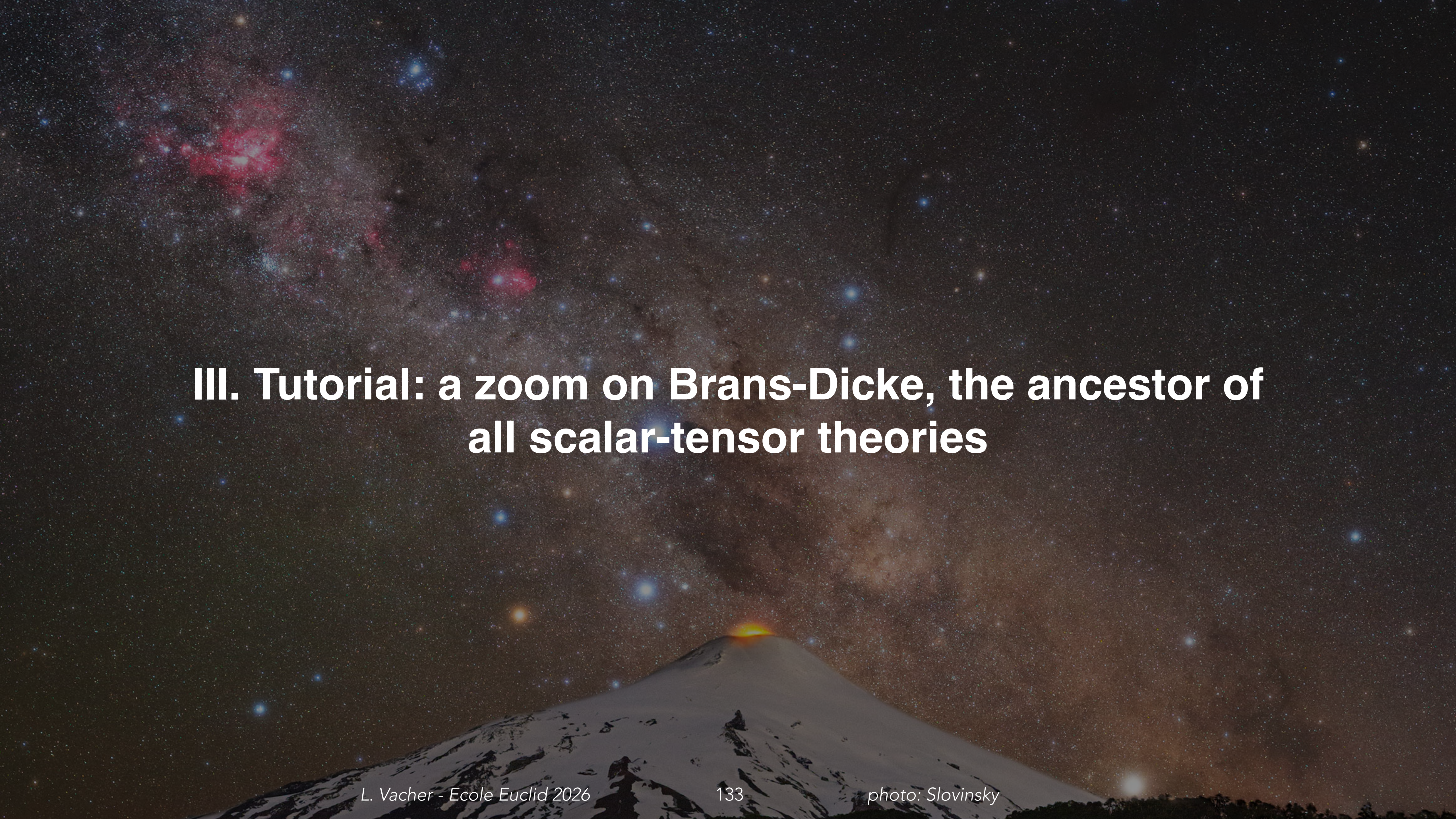
AND SO ON



# Conclusions

- GR is **rich conceptually** and its worth understanding all its details before going beyond
- GR has been validated **extremely well** in all but quantum regimes
- possibilities for modified gravity models are limited by **no-go theorems**
- Yet there is a lot of them with **different motivations!**
- Few deep and basic principles allows you to get a **general view**
- They can be probed by **cosmology** and a lot is yet to come
- We **barely scratched the surface!** Still, I hope you can see a bit better the **landscape**



The background of the slide is a composite image. At the bottom, a snow-covered mountain peak is visible, with a small, bright yellow light source (possibly a setting or rising sun or moon) just above its summit. The sky above is filled with stars and the Milky Way galaxy, which appears as a dense band of light. A prominent red nebula is visible in the upper left quadrant of the sky.

# **III. Tutorial: a zoom on Brans-Dicke, the ancestor of all scalar-tensor theories**



# Mach's principle

What is the origin of inertia?

$$-\frac{GMm}{R} \simeq mc^2 \quad \longrightarrow \quad -\frac{GM}{Rc^2} \simeq 1 \quad \longrightarrow \quad \frac{1}{G} \simeq \frac{M}{Rc^2}$$

## Brans-Dicke Lagrangian and equations of motion

$$\mathcal{L} = \frac{1}{16\pi} \left( \phi R - \frac{\omega}{\phi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right) + \mathcal{L}_m[\psi, g],$$



# Brans-Dicke Lagrangian and equations of motion

**Proove that:**

$$G_{\mu\nu} = \frac{8\pi}{\phi} T_{\mu\nu} + \frac{\omega}{\phi^2} (\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi) + \frac{1}{\phi} (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \square \phi)$$

**Bonus:**

$$\delta \sqrt{-|g|} = -\frac{1}{2} \sqrt{-|g|} g_{\mu\nu} \delta g^{\mu\nu}$$

$$\phi \frac{\delta R}{\delta g^{\mu\nu}} = \phi R_{\mu\nu} + g_{\mu\nu} \square \phi - \nabla_\mu \nabla_\nu \phi$$

# Brans-Dicke Lagrangian and equations of motion

**Proove that:**

$$\square \phi = \frac{8\pi}{3 + 2\omega} \mathcal{T}$$

**Bonus:**

Re-use Einstein equations!



# Brans-Dicke Lagrangian and equations of motion

**Proove that:**

$$\nabla^{\mu} T_{\mu\nu} = 0$$

**Bonus:**

Use Bianchi identity  $\nabla^{\mu} G_{\mu\nu} = 0$

# Einstein frame

**Proove that:**

$$S = \int d^4x \sqrt{-|\tilde{g}|} \left[ \frac{\tilde{R}}{16\pi G_0} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} \right] + S_m \left[ e^{-\kappa \tilde{\phi}} \tilde{g}_{\mu\nu}, \psi \right], \quad \kappa \equiv \sqrt{\frac{16\pi G_0}{2\omega + 3}}.$$

**With:**

$$\tilde{\phi} = \sqrt{\frac{2\omega + 3}{16\pi G_0}} \ln \frac{\phi}{\phi_0}$$

**Bonus:**

$$R = \Omega^2 \left[ \tilde{R} + 6 \tilde{\square} \ln \Omega - 6 (\tilde{\nabla} \ln \Omega)^2 \right].$$

$$g_{\mu\nu} = \Omega^{-2} \tilde{g}_{\mu\nu}, \quad \Omega^2 \equiv \frac{\phi}{\phi_0}$$



**Conformal models do not affect light!  $\Sigma = 0$**

**PROOVE IT!**

## Equations of motion in the Einstein frame

$$\tilde{G}_{\mu\nu} = 8\pi G_0 \left[ \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{g}^{\alpha\beta} \partial_\alpha \tilde{\phi} \partial_\beta \tilde{\phi} + \tilde{T}_{\mu\nu} \right]$$

$$\tilde{\square} \tilde{\phi} = \frac{\kappa}{2} \tilde{\mathcal{T}}$$

$$\tilde{\nabla}^\mu \tilde{T}_{\mu\nu} = -\frac{\kappa}{2} \tilde{\mathcal{T}} \partial_\nu \tilde{\phi}$$

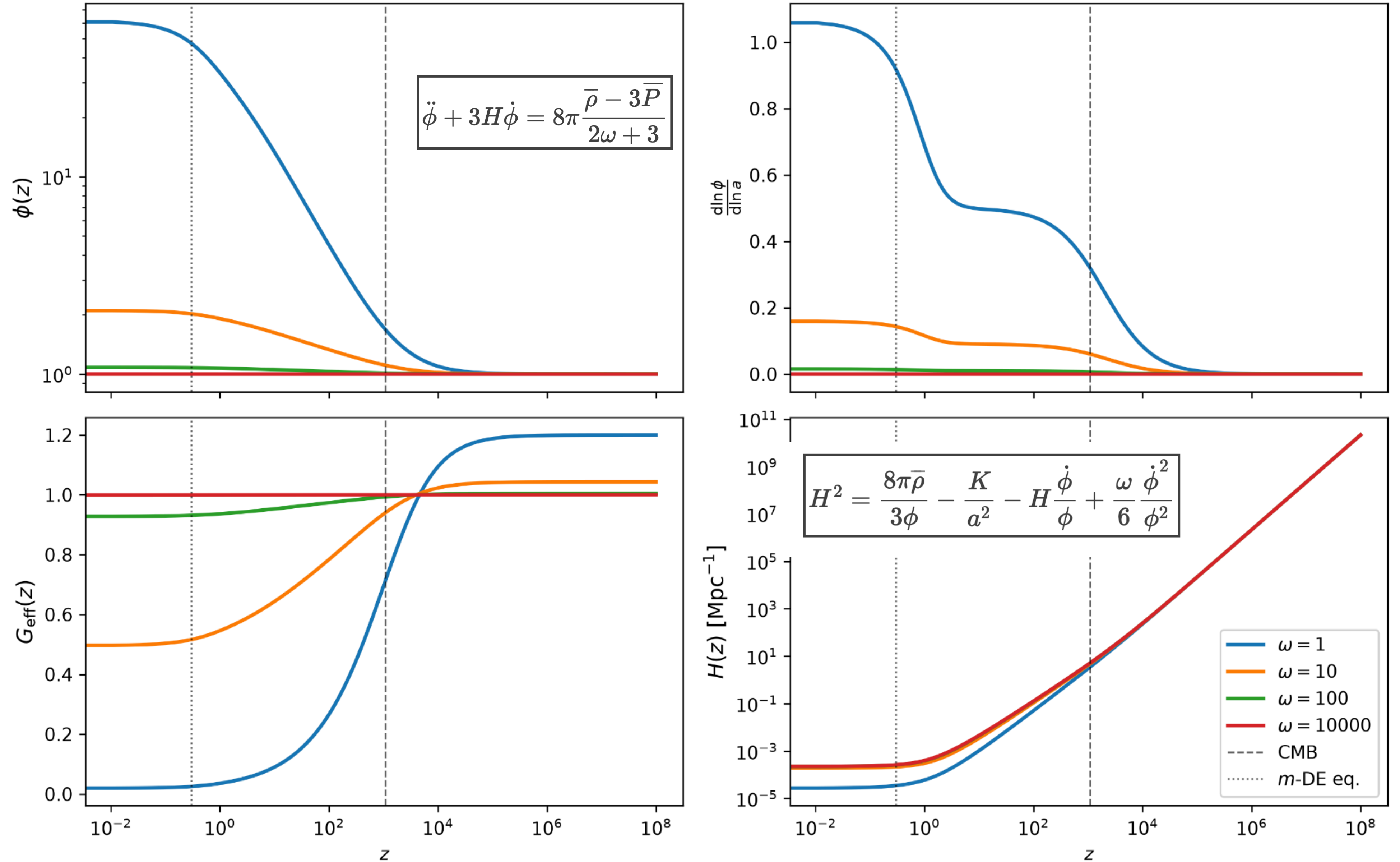


# Cosmological evolution

$$H^2 = \frac{8\pi\bar{\rho}}{3\phi} - \frac{K}{a^2} - H\frac{\dot{\phi}}{\phi} + \frac{\omega}{6}\frac{\dot{\phi}^2}{\phi^2}$$

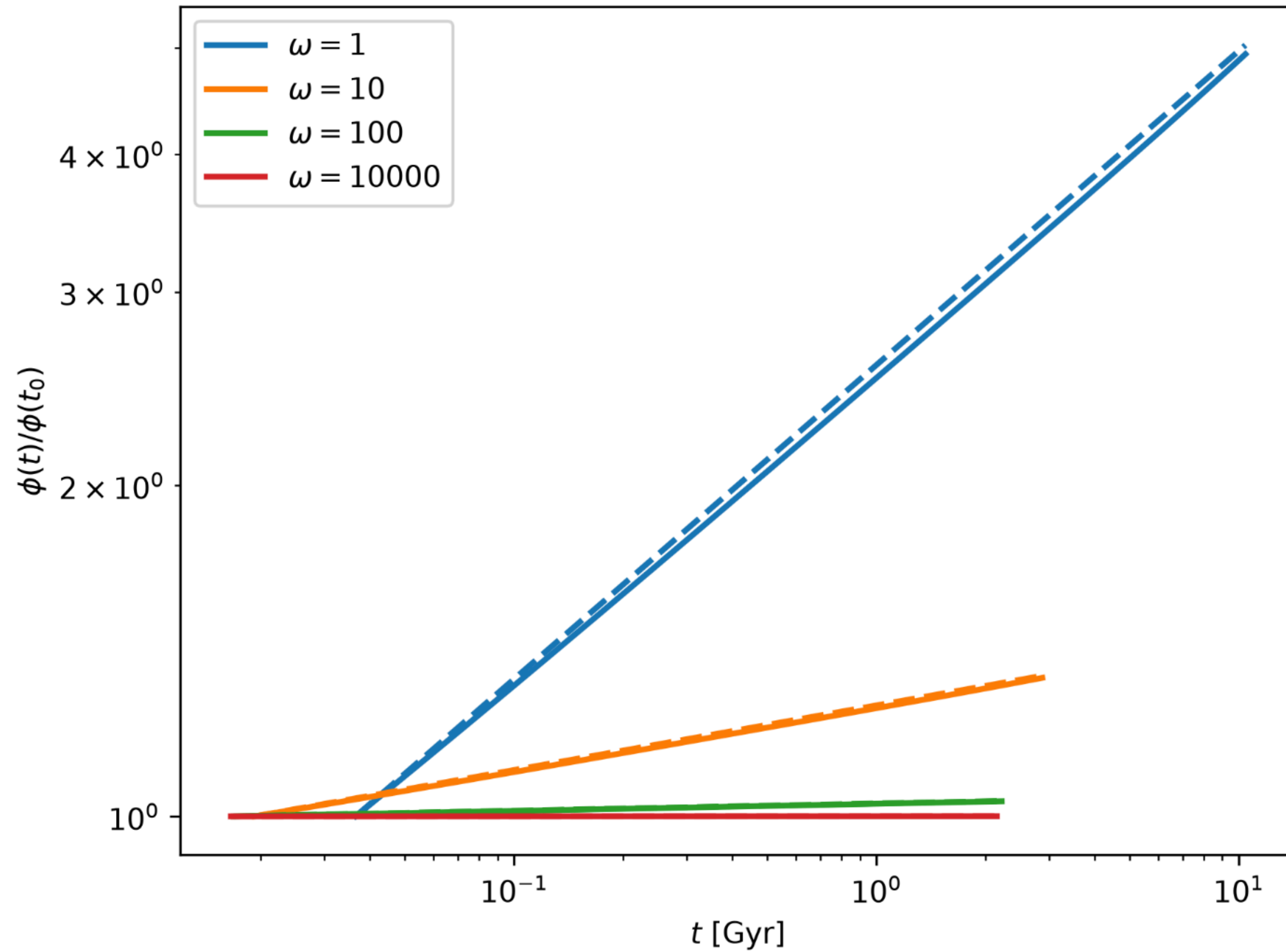
$$\ddot{\phi} + 3H\dot{\phi} = 8\pi\frac{\bar{\rho} - 3\bar{P}}{2\omega + 3}$$

# Cosmological evolution





# Cosmological evolution



## Brans-Dicke with a potential

$$H^2 = \frac{8\pi\bar{\rho}}{3\phi} - H\frac{\dot{\phi}}{\phi} + \frac{\omega}{6}\frac{\dot{\phi}^2}{\phi^2} + \frac{8\pi}{3}\frac{V(\phi)}{\phi}$$

$$\ddot{\phi} + 3H\dot{\phi} = \frac{8\pi}{2\omega + 3} \left( \bar{\rho} - 3\bar{P} + 4V(\phi) - 2\frac{dV}{d\ln(\phi)} \right)$$



## Varying G

$$G_{\text{eff}} = \frac{1}{\phi_0} \cdot \frac{2\omega + 4}{2\omega + 3}$$

$$|\dot{G}/G| = (-5.0 \pm 9.6) \times 10^{-15} \text{ yr}^{-1} \text{ (LLR)}$$

# Solar system constraints

